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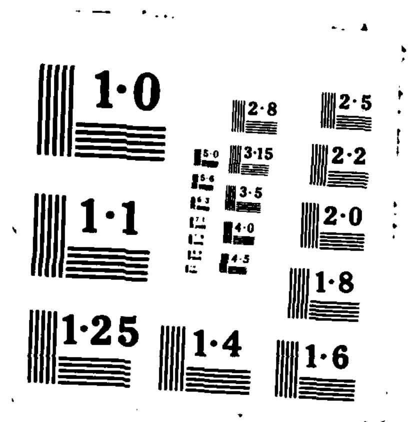
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A DISCRETIZATION OF THE INTEGRAL EQUATION FOR THE TIME DEPENDENT LINEARIZED SUBSONIC POTENTIAL FLOW OVER A WING

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In a previous report the integral equation for linearized time dependent subsonic potential flow over a wing has been derived. The report describes how this formulation can be discretized. The wing and the wake are divided into triangles. The flow is described by the time dependent potential at the corners of the triangles; in other words the program generates for each corner a table of the potential versus time. The interpolations necessary for the integrations between the corners points and also with respect to time, are described in detail. Initially the upwash is assumed to be known as a function of time over the planform. In practice one will express the upwash by a rather limited number of standard functions defined over the planform multiplied by a superposition of step functions in time. Then it suffices to determine for each of the spatial standard functions the response to a step function in time. In this form the method can be used in aeroelastic problems where the time dependence of the deformations is not always known in advance. In such problems overall forces and momenta rather than the detailed pressure distributions are needed.

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This is discussed in some detail. After the initial response to a step function the flowfield is rather smooth (although still time dependent). Therefore it can be described by a reduced number of parameters and one can proceed in greater time steps. The procedure by which this can be done is also described.

PREFACE

The work was performed during the period May 1986 through January 1987 under contract F33615-86-3200 entitled "A Study of Linearized Integral Equation for Steady and Oscillatory Supersonic Flow" to the University of Dayton for the Aeroelastic Group, Analysis and Optimization Branch, Structures Division, Air Force Wright Aeronautical Laboratories, Air Force Systems Command under Program Element 61102F, Project No. 2304, Task N1, Work Unit 22. Dr Karl K. Guderley of the University of Dayton Research Institute was the Principal Investigator. Dr. Charles Keller, AFWAL/FIBRC was Program Manager.

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INTRODUCTION

The equation for three dimensional subsonic unsteady potential flow possesses fundamental solutions in a closed form. Therefore, one can readily bring the problem of the flow over a wing into the form of an integral equation with two independent variables. This formulation is, however, not directly suitable for numerical evaluation, for it requires that one carry out a limiting process in which one approachs the planform from above or below. In Reference 1 transformations have been carried out which generate a form in which all expressions have a direct meaning within the planform of the wing.

In these derivations no simplifications (except for the initial assumption of a linearized flow) have been made. For numerical purposes a discretization must be carried out. The present report describes a possible procedure of this kind. Most of the report deals with the discretization process. A preliminary code for the two-dimensional case is found in Reference 2.

The method proposed here is rather versatile but time consuming. In order to economize one must know which information is required in the setting in which the results will be used. The author has, therefore, added some sections which deal in general terms with the form in which these results will be used in aeroelasticity. This includes an attempt to characterize the necessary time resolution, and a method of reducing the size of the problem after initial "irregularities" have died out.

SECTION II

COMPILATION OF THE RESULTS OF REFERENCE 1

The results obtained in Reference 1 are repeated here, arranged in a form better suited for the numerical work. In Reference 1 the Prandtl-Glauert coordinate distortion has been introduced. For unsteady flows it is less useful than for steady flows because it obscures to some extent the manner in which perturbations propagate; moreover the effect of sweep-back is more clearly expressed without this distortion. We shall go to the undistorted (but dimensionless) variables in Section III. To prepare for it we characterize variables in the distorted system (which in Reference 1 appear without tilde) by a tilde. This will allow us to use variables without tilde through the undistorted system.

In this section we compile the results in the distorted system of coordinates. In Reference 1 we denoted by

U = free-stream velocity

$\bar{a}$ = free-stream velocity of sound

$M = U/\bar{a}$ = free-stream Mach number

$\bar{\rho}$ = free-stream density

L = a characteristic length

The coordinates in physical space and the time are denoted by $\bar{x}$, $\bar{y}$, $\bar{z}$, and $\bar{t}$. The $\bar{x}$ direction is the free-stream direction, the planform of the wing lies in the $\bar{x},\bar{y}$-plane. Let $\phi(\bar{x},\bar{y},\bar{z},\bar{t})$ be the potential and $\bar{p}$ the perturbation pressure. Dimensionless quantities and the Prandtl-Glauert distortion are introduced by

$$\tilde{x} = \bar{x}/L$$
$$\tilde{y} = (1 - M^2)^{1/2}\bar{y}/L$$
$$\tilde{z} = (1 - M^2)^{1/2}\bar{z}/L$$

$$\tilde{t} = (\bar{a}\bar{t})/L$$

$$\tilde{\phi}(x,y,z,t) = UL\bar{\phi}(\tilde{x}L,(1 - M^2)^{-1/2}\tilde{y}L,(1 - M^2)^{-1/2}\tilde{z}L,\tilde{t}L/a)$$

$$\tilde{p} = \bar{p}/\bar{\rho}U^2$$

Let $\bar{x} = \bar{x}_{tr}(\bar{y})$ be the equation of the trailing edge. Then

$$\tilde{x} = \tilde{x}_{tr}(\tilde{y})$$

with

$$\tilde{x}_{tr}(\tilde{y}) = L^{-1}\bar{x}_{tr}((1 - M^2)^{-1/2}\tilde{y}L)$$

Let $\bar{z} = \bar{g}(\bar{x},\bar{y},\bar{t})$ give the time history of the displacement or deformations of the wing surface, and

$$\tilde{g}(\tilde{x},\tilde{y},\tilde{t}) = L^{-1}\bar{g}(\tilde{x}L,(1 - M^2)^{-1/2}\tilde{y}L,\tilde{t}L\bar{a}^{-1})$$

To express the boundary conditions at the wing, Reference 1 introduces

$$\tilde{w}_o(\tilde{x},\tilde{y},\tilde{t}) = (1 - M^2)^{-1/2}(\tilde{g}_{\tilde{x}} + M^{-1}\tilde{g}_{\tilde{t}}) \tag{1}$$

The dependent variable in the problem is $\tilde{h}(\tilde{x},\tilde{y},\tilde{t})$. One has

$$\tilde{\phi}(\tilde{x},\tilde{y},\tilde{z}=\pm 0,\tilde{t}) = \pm 2\pi\tilde{h}(\tilde{t},\tilde{x},\tilde{y})) \tag{2}$$

The notation $\tilde{z} = \pm 0$ means that one approaches $\tilde{z} = 0$ from positive or negative values. The values of $\tilde{h}$ in the wake are expressed by its values at the trailing edge. To emphasize this we introduce

$$\tilde{h}_{tr}(\tilde{t},\tilde{y}) = \tilde{h}(\tilde{t},\tilde{x}_{tr}(\tilde{y}),\tilde{y})$$

Then

$$\tilde{h}(\tilde{t},\tilde{x},\tilde{y}) = h_{tr}(t - M^{-1}(\tilde{x} - \tilde{x}_{tr}),\tilde{y})) \qquad (3)$$

In Reference 1 the notation $\tilde{h}^{(1)}$, $\tilde{h}^{(2)}$, and $\tilde{h}^{(3)}$ has been introduced for the derivative of $\tilde{h}$ with respect to its first, second, and third argument.

In the integro-differential equation for $\tilde{h}$, $\tilde{x}$ and $\tilde{y}$ are points of the wing areas, the umbral variables $\tilde{\xi}$ and $\tilde{\eta}$ range over the wing and the wake. For each choice of the point $\tilde{x},\tilde{y}$ the integration region is divided into an area surrounding the point $\tilde{x},\tilde{y}$ denoted by A_1, and the remaining area (wing area + wake - A_1) denoted by A_2. The contour of A_1 is denoted by Γ_1. One has the following auxiliary expression

$$\tilde{\rho} = [(\tilde{\xi} - \tilde{x})^2 + (\tilde{\eta} - \tilde{y})^2]^{1/2}$$

$$\tilde{ret} = (1 - M^2)^{-1}[M(\tilde{\xi} - \tilde{x}) + \tilde{\rho}]$$

$$\tilde{\tau} = \tilde{t} - \tilde{ret}$$

$$\tilde{f}^{(2)} = \frac{(\tilde{\xi} - \tilde{x}) + M\tilde{\rho}}{\tilde{\rho}[M(\tilde{\xi} - \tilde{x}) + \tilde{\rho}]^2}$$

$$\tilde{f}^{(3)} = \frac{(1 - M^2)(\tilde{\eta} - \tilde{y})}{\tilde{\rho}[M(\tilde{\xi} - \tilde{x}) + \tilde{\rho}]^2}$$

Then one finds in Reference 1 the following integro-differential equation for h.

$$\tilde{w}_o(\tilde{x},\tilde{y},\tilde{t}) = -2\pi(1 - M^2)^{-1/2}[\tilde{h}^{(1)}(\tilde{t},\tilde{x},\tilde{y}) + M\tilde{h}^{(2)}(\tilde{t},\tilde{x},\tilde{y})] + \tilde{B}_7 + \tilde{B}_8 + \tilde{B}_9 \qquad (4)$$

where

$$\tilde{B}_7 = \iint_{A_2} \{\tilde{f}^{(2)}\tilde{h}^{(2)}(\tilde{\tau},\tilde{\xi},\tilde{\eta}) + \tilde{f}^{(3)}\tilde{h}^{(3)}(\tilde{\tau},\tilde{\xi},\tilde{\eta})\}d\tilde{\xi}d\tilde{\eta}$$

$$B_8 = \tilde{h}_2^{(2)}(\tilde{t},\tilde{x},\tilde{y}) \oint_{\Gamma_1} -(\tilde{\eta} - \tilde{y})\frac{(\tilde{\xi} - \tilde{x} + M\tilde{\rho})d\tilde{\xi} + (\tilde{\eta} - \tilde{y})d\tilde{\eta}}{\tilde{\rho}[M(\tilde{\xi} - \tilde{x}) + \tilde{\rho}]^2}$$

$$+ \tilde{h}_3(\tilde{t},\tilde{x},\tilde{y}) \oint_{\Gamma_1} (\tilde{\xi} - \tilde{x} + M\tilde{\rho})\frac{(\tilde{\xi} - \tilde{x} + M\tilde{\rho})d\tilde{\xi} + (\tilde{\eta} - \tilde{y})d\tilde{\eta}}{\tilde{\rho}[M(\tilde{\xi} - \tilde{x}) + \tilde{\rho}]^2}$$

$$B_9 = \iint_{A_1} \{\tilde{f}^{(2)}(\tilde{h}^{(2)}(\tilde{\tau},\tilde{\xi},\tilde{\eta}) - \tilde{h}^{(2)}(\tilde{t},\tilde{x},\tilde{y}))$$

$$+ \tilde{f}^{(3)}(\tilde{h}^{(3)}(\tilde{\tau},\tilde{\xi},\tilde{\eta}) - \tilde{h}^{(3)}(\tilde{t},\tilde{x},\tilde{y}))\}d\tilde{\xi}d\tilde{\eta}$$

$\tilde{B}_7$ and $\tilde{B}_9$ are obviously closely related to each other, but $\tilde{B}_7$ does not converge in a region containing the point $\tilde{x},\tilde{y}$. For details see Reference 1.

If $\tilde{h}(\tilde{t},\tilde{x},\tilde{y})$ has been determined up to a time $\tilde{t}$ for all points within the planform (and because of Eq. (3) also within the wake), then Eq. (4) allows one to evaluate $\partial\tilde{h}/\partial\tilde{t} = \tilde{h}^{(1)}(\tilde{t},\tilde{x},\tilde{y})$ and then proceed by another time step. After $\tilde{h}$ has been found, one evaluates $\tilde{\phi}(\tilde{x},\tilde{y},\tilde{z} = +0,\tilde{t})$, Eq. (2), and the dimensionless deviation from the free-stream pressure

$$\tilde{p} = -(M^{-1}\tilde{\phi}_{\tilde{t}} + \tilde{\phi}_{\tilde{x}})$$

and the actual deviation from the free-stream pressure

$$\bar{p} = \bar{\rho}U^2\tilde{p}$$

The factor M^{-1} in the formula for $\tilde{p}$ occurs because the time $\tilde{t}$ and $\tilde{\phi}$ have been made dimensionless with the velocity of sound $\bar{a}$ and the free-stream velocity U, respectively. (If one makes $\tilde{t}$ dimensionless with $\bar{a}$, then a perturbation travels by the characteristic length L during a time interval 1 measured in $\tilde{t}$.

If one makes $\bar{t}$ dimensionless with U, then a particle moves by the characteristic length L during the interval 1 measured in $\bar{t}$. Both definitions have a sound physical meaning.)

SECTION III

FORMULATION IN AN UNDISTORTED COORDINATE SYSTEM

One returns to an undistorted coordinate system (x,y,z,t) by the transformation

$$\tilde{x} = \bar{x}/L = x$$

$$\tilde{y} = (1 - M^2)^{1/2}\bar{y}/L = (1 - M^2)^{1/2}y$$

$$\tilde{z} = (1 - M^2)^{1/2}\bar{z}/L = (1 - M^2)^{1/2}z$$

$$\tilde{t} = a\bar{t}/L = t$$

$$\tilde{\xi} = \xi$$

$$\tilde{\eta} = (1 - M^2)^{1/2}\eta$$

$$\phi(x,y,z,t) = \tilde{\phi}(x,(1 - M^2)^{1/2}y,(1 - M^2)^{1/2}z,t)$$

The basic data in physical space are, of course, the same. The time history of the displacements or deformation is given by

$$\bar{z} = \bar{g}(\bar{x},\bar{y},\bar{t})$$

Hence, in undistorted coordinates

$$z = g(x,y,t) = L^{-1}\bar{g}(Lx,Ly,Lt/a)$$

We introduce

$$w_o(x,y,t) = g_x + M^{-1}g_t = (1 - M^2)^{1/2}\tilde{w}_o(x,(1 - M^2)^{1/2}y,t)$$

The dependent variable $\tilde{h}(\tilde{t},\tilde{x},\tilde{y})$ is replaced by

$$h(t,x,y) = \tilde{h}(t,x,(1 - M^2)^{1/2}y)$$

We denote by $h^{(1)}$, $h^{(2)}$, and $h^{(3)}$, respectively, the derivative of h with respect to its first, second, and third argument.

$$h^{(1)} = \tilde{h}^{(1)}$$

$$h^{(2)} = \tilde{h}^{(2)}$$

$$h^{(3)} = (1 - M^2)^{1/2}\tilde{h}^{(3)} \quad , \quad \tilde{h}^{(3)} = (1 - M^2)^{-1/2}h^{(3)}$$

One has the relation

$$\phi(x,y,z=\pm 0,t) = \pm 2\pi h(t,x,y)$$

The shape of the trailing edge

$$\bar{x} = \bar{x}_{tr}(\bar{y})$$

is now given by

$$x = x_{tr}(y) = L^{-1}\bar{x}_{tr}(Ly)$$

We introduce again

$$h_{tr}(t,y) = h(t,x_{tr}(y),y)$$

Then one has within the wake

$$h(t,x,y) = h_{tr}(t - M^{-1}(x - x_{tr}(y),y))$$

One needs the following auxiliary functions

$$\rho = [(\xi - x)^2 + (1 - M^2)(\eta - y)^2]^{1/2}$$

$$\text{ret} = (1 - M^2)^{-1}[M(\xi - x) + \rho]$$

$$\tau = t - \text{ret}$$

$$f^{(2)} = \frac{\xi - x + M\rho}{\rho[M(\xi - x) + \rho]^2}$$

$$f^{(3)} = \frac{(1 - M^2)(\eta - y)}{\rho[M(\xi - x) + \rho]^2}$$

The expression $f^{(3)}$ differs from the expression resulting from substitution into $\tilde{f}^{(3)}$ by a factor $(1 - M^2)^{1/2}$. After multiplication by $(1 - M^2)^{1/2}$, one then obtains

$$w_o(x,y,t) = -2\pi[h^{(1)}(t,x,y) + Mh^{(2)}(t,x,y)] + B_7 + B_8 + B_9 \tag{5}$$

where

$$B_7 = (1 - M^2)\iint_{A_2} f^{(2)}h^{(2)}(\tau,\xi,\eta) + f^{(3)}h^{(3)}(\tau,\xi,\eta)d\xi d\eta$$

$$B_8 =$$

$$\tag{6}$$

$$(1 - M^2)h^{(2)}(t,x,y)\oint_{\Gamma_1} - (\eta - y)\frac{(\xi - x + M\rho)d\xi + (1 - M^2)(\eta - y)d\eta}{\rho[M(\xi - x) + \rho]^2}$$

$$+ h^{(3)}(t,x,y)\oint_{\Gamma_1}(\xi - x + M\rho)\frac{(\xi - x + M\rho)d\xi + (1 - M^2)(\eta - y)d\eta}{\rho[M(\xi - x) + \rho]^2}$$

$$B_9 = (1 - M^2)\iint_{A_1} \{f^{(2)}(h^{(2)}(\tau,\xi,\eta) - h^{(2)}(t,x,y))$$

$$+ f^{(3)}(h^{(3)}(\tau,\xi,\eta) - h^{(3)}(t,x,y))\}d\xi d\eta$$

After h(t,x,y) has been found, one determines

$$\phi(t,x,y,z=+0,t) = 2\pi h(t,x,y)$$

$$p = -(M^{-1}\phi_t + \phi_x)$$

and

$$\bar{p}(\bar{x},\bar{y},\bar{t}) = \rho U^2 p(\frac{\bar{x}}{L}, \frac{\bar{y}}{L}, \frac{\bar{t}a}{L})$$

The problem will be discussed in the form shown here. In the computation we evaluate for each time step $h^{(1)} = \frac{\partial h}{\partial t}$. The equation is therefore used in the form

$$\frac{\partial h}{\partial t}(t,x,y) = -(2\pi)^{-1} w_o(x,y,t) + Mh_x(t,x,y)$$

$$+ (2\pi)^{-1}[B_7 + B_8 + B_9]$$

SECTION IV

SOME GENERAL REMARKS

It is desirable to hold the number of parameters by which the wing potential is described small. In the present approach, this means that the number of points within the planform for which the potential is determined should not be excessive. A lower limit for the number of points is, however, unavoidable. By discretizing in space one also restricts the resolution in time, because the method is unable to follow the propagation of pressure waves through the individual surface elements. As a consequence, the high frequency components of the motion are lost, more so in a coarser than in a finer grid. A minimum number of subdivisions of the surface is, therefore, necessary.

At the leading edge, the potential shows a singular behavior. In a steady flow it behaves as $u^{1/2}$ multiplied by a constant where u is the distance from the leading edge. The pressures then behave as const times $u^{-1/2}$. If one approximates the potential at the leading edge in a simple manner, namely by a linear function, then the pressure in this region will be constant. This by itself is not objectionable; in the interior of the wing such an approximation is sufficient. But experience has shown that the integrated pressures (the contributions to the total lift and moment) for the mesh elements at the leading edge are very inaccurate. One might remedy this by a correction factor (universal for all elements at the leading edge and restricted to this area). This possibility might be worth exploring, it might lead to a simplification of the programming work. In the present report the author has chosen to approximate the potential in the vicinity of the leading edge by a function which behaves as $u^{1/2}$ and thus to anticipate the behavior for the steady case. Details will appear later.

In the present report the analysis is restricted to the wing. This should be taken as the first step towards an approach in which wing and fuselage are taken into account simultaneously.

For a slender fuselage a linearized approach is again applicable. The treatment of the fuselage is simpler than the treatment of the wing, at least in principle, for the potential can be expressed by a distribution of sources over its surface rather than by doublets, and the singularities in the kernel of the integral equation are less severe. Theoretically, one could apply such a treatment also to the wing. Then one would not reduce it to its planform but consider it as three-dimensional and satisfy the boundary conditions separately at the upper and lower surfaces. The treatment in terms of doublets is, however, preferable; the upper and lower wing surfaces are close to each other. As a consequence of the strong interference between sources at the two surfaces the result will appear as the difference between large numbers. This difficulty vanishes if one uses doublets rather than single poles at the wing surface with opposite signs. A method which uses single poles at the fuselage and at the faired-in transition from the fuselage to the wing and doublets at the planform can probably be developed without difficulties. In each representation one has to express the velocity component normal to the planform and to the fuselage in terms of the unknown parameters describing the doublet and source strengths. From these data the potential can be computed.

If one disregards the fuselage one is inclined to take the symmetry of the problem into account by considering a planform as shown in Figure 1. This, however, generates a difficulty at point A if the wing has sweep. There the flow pattern becomes very complicated. The $u^{1/2}$ singularity of the potential at the leading edge still persists, but no analytical solution for the interior is available which could be used in the computation to anticipate the local character of the flow field. To avoid this difficulty, we shall consider a planform as shown in Figure 2.

In the vicinity of the wing root an approach which replaces a wing by its planform is unsatisfactory in any case. The fairing in between wing and fuselage not taken into account at

all and the effect of the fuselage on the wing is modeled only approximately. Therefore, it does not really matter, whether one treats the problem of Figure 1 or Figure 2. A satisfactory treatment will always require a theory in which the fuselage is included.

SECTION V

THE SOLUTION PROCESS DESCRIBED IN GENERAL TERMS

The main task is the evaluation of the integrals B_7, B_8, and B_9. To carry out the discretization needed for this purpose the author proposes to divide the planform into triangular subareas. The unknown function h will be characterized by its values at the corner points of the triangles. For the interior of the triangles an interpolation will be used. The corners are numbered by two indices (i and j) roughly corresponding to the numbering scheme suggested by a two-dimensional coordinate system. The program will generate for each of these corners a table which gives the values of h at evenly spaced times. It is assumed that the same time interval Δt is maintained throughout the computation. The entries within these tables are indexed by a subscript k (which then refers to a time $k\Delta t$). In the memory the function h then appears as a subscripted variable with three subscripts, the first two for the corner point on the planform and the last one for the time.

Assume that at some stage in the computation these tables have been generated up to a time index k. The above equations then allow one to evaluate $\partial h/\partial t$ at this time for all stations. These values are then used to proceed by one time step. The present report deals with the evaluation of $\partial h/\partial t$. In evaluating $\partial h/\partial t$ at a point i,j one has to carry out an integration over the wing and the wake area. This integration appears then as a summation over the points of the wing and wake area. The summation subscripts will be m and n. (The distinction between i,j on one hand and m,n on the other is analogous to the distinction between the coordinates x and y for which the integro-differential equation is to be satisfied and the umbral variables of integration ξ and η which are found in the integrals.) The boundary condition, namely the upwash at the point i,j enter in the expression [illegible] through w. The point i,j plays a role similar [illegible] methods

for steady or oscillatory flows; occasionally we shall use this term.

After the discretization the integro-differential equation will assume the form

$$\frac{\partial h_{ijk}}{\partial t} = -(2\pi)^{-1} w_{o,ijk}$$

$$+ \sum_{mn\ell} (C_{(ij),(mn),\ell}\, h_{mn,(k-\ell)}) \qquad \text{for all } ij \quad (7)$$

Here $w_{o,ijk}$ is the value of the prescribed function w_o at the control point ij and at the time pertaining to the index by k. The first two subscripts of h on the left refer to the control point; the third one to the current time $k\Delta t$. The first two subscripts of h in the sum on the right refer to the field point in question; the last subscript $(k - \ell)$ to the current time retarded by $\ell\Delta t$. The term $\partial h/\partial x$ which also appears in the basic equation is incorporated in the sum. Eq. (7) simply means that the time derivative of h at each point i,j depends linearly upon the values of h at the points of the wing (and of the wake) at times retarded by suitable amounts. The coefficients $C_{(ij),(mn),\ell}$ are constants. In this report we shall show how these coefficients can be evaluated.

The fact that the summation in Eq. (7) is carried out over three subscripts (m,n and ℓ) while the integrals in Eq. (6) have only the two umbral variables ξ and η, requires an explanation. The summation over m and n corresponds to the integration over ξ and η. In the integrands the functions $h^{(2)}$ and $h^{(3)}$ occur with a retarded time as argument. This explains the fact that the time index in the sums is not k (the current time) but $k - \ell$. For each choice of the control point ij and of a point $\xi\eta$ within the field the retardation has a specific value expressed by $\ell\Delta t$. (We allow temporarily ℓ to be different from an integer.) The

retardation is not quite constant because the point $\xi\eta$ in the field can be regarded as a substitute for a whole area surrounding it.

Assume that for fixed ij and fixed mn only one value of the retardation, denoted by ℓ_o, is obtained. Then Eq. (7) would still be applicable but all coefficients of $C_{(ij),(mn),\ell}$ would be zero except for $\ell = \ell_o$. However, this is not the case because mn stands for an area in the $\xi\eta$ plane. Moreover, an interpolation with respect to time is needed to accommodate nonintegral values of the retardation . One, therefore, will obtain a band (in terms of ℓ) of the values $C_{(ij),(mn),\ell}$ in which these quantities are different from zero; this band is fairly narrow.

The coefficients $C_{(ij),(mn),\ell}$ arise in essence by evaluating the integrals B_7, B_8, or B_9 at fixed ij, separately for the individual triangular areas or for the hexagon surrounding the point ij. Adjacent to each point (mn) there are six triangles. Within the triangles h is determined by its values at the corners. Accordingly, the integrands for one triangle of the hexagon depends upon the values of h at the corner points. To be more specific one evaluates in each area the integrands at certain pivotal points and multiplies them by a factor which is roughly speaking the equivalent of the area pertaining to the pivotal point. The contribution of one pivotal point depends upon the value of $h^{(2)}$ and $h^{(3)}$ and, therefore, upon the values of h at the corners of the area. One obtains contributions of each pivotal point to the values of $C_{(ij),(mn),\ell}$ for those values of mn which determine h within the area. After the procedure has been carried out for all ij and for all pivotal points on the wing and the wake, the evaluation of the coefficients C_{ijmn} is complete. The report discusses in detail:

1. The numbering system for the corners of the triangles.

2. The integration procedure in general.

3. The representation of h within the triangles and within the areas surrounding a control point, and transformations required to obtain well behaved integrands.

4. A criterion to decide in a given region which precision of the integration formulae is needed. (Such a criterion would replace a decision made otherwise by inspections.)

SECTION VI

INDEXING OF CORNERS AND OF AREAS

The numbering scheme is obtained in the following manner. The planform is first divided into quadrangular areas. Triangles are obtained by drawing one diagonal within each quadrangle. The corners of the triangles are, of course, the same as the corners of the quadrangles. The corners are characterized by two indices, the first one for the (approximate) chord direction, the second for the span direction. The subscripts used for a control point are i and j, for points within the field m and n. The value of i for the leading edge is zero, for the trailing edge i_t. The quadrangles are characterized by the indices at their lower left-hand corner. The two triangles which arise after drawing one diagonal within the quadrangle are distinguished by further subscripts 1 or 2 (Figure 3). It may appear clumsy to use three subscripts to identify one triangle but the memory space thus taken up is not larger than the memory space for a numbering by one index.

The geometry of the problem is introduced into the computation by a list of the corner coordinates x_{ij} and y_{ij}. If the wing is symmetric, one need only consider the right-half of the wing. The x axis is then the line of symmetry. The symmetric and antisymmetric parts of the motions and deformations of the wing would be treated separately. We write down the corner numbers pertaining to certain areas.

While one carries out computations for a specific triangle, the indices of the corners are denoted by $\bar{m}_\ell$ and $\bar{n}_\ell$. One has for a triangle (mn1)

$$\begin{aligned} \bar{m}_1 &= m \quad , \quad \bar{n}_1 = n \\ \bar{m}_2 &= m + 1 \quad , \quad \bar{n}_2 = n \\ \bar{m}_3 &= m \quad , \quad \bar{n}_3 = n + 1 \end{aligned} \tag{8}$$

(The values of h, ξ, and η for these three points will be denoted by $\bar{h}_\ell$, ξ_ℓ, and η_ℓ; ℓ = 1, 2, 3. The symbol $\bar{h}$ instead of h is introduced because h appears already as subscripted variable h_{mnk} in the program. The above relation (Eq. (8)) will occur only twice in the treatment of a specific triangle, the first time when one identifies the coordinates of the corner points, for instance,

$$\bar{\xi}_1 = \xi_{\bar{m}(1),\bar{n}(1)}$$

the second time after the integration for the area has been carried out, when one identifies $\bar{h}_\ell$ with $h_{\bar{m}(\ell),\bar{n}(\ell)}$*. Therefore there is no need to introduce the $\bar{m}_\ell$ and $\bar{n}_\ell$ in the program explicitly. Furthermore, one has for a triangle (mn2)

$$\begin{aligned} \bar{m}_1 &= m \quad , \quad \bar{n}_1 = n + 1 \\ \bar{m}_2 &= m + 1 \quad , \quad \bar{n}_2 = n \\ \bar{m}_3 &= m + 1 \quad , \quad \bar{n}_3 = n + 1 \end{aligned} \tag{9}$$

A control point ij appears as vertex in six triangles surrounding it. For the whole area covered by these six triangles a single representation for h is used; it depends upon the values of h at the corners of all these triangles, including, of course, h_{ij}.

---

*In these expressions we have written $\bar{m}(\ell)$ instead of $\bar{m}_\ell$ in order to avoid two level subscripts. In general we shall prefer notations by subscripts.

While one deals with the area surrounding a specific control point ij, these points are numbered from 1 to 7 (Figure 4). One has

$$\begin{aligned}
\bar{m}_1 &= i &, \quad \bar{n}_1 &= j \\
\bar{m}_2 &= i &, \quad \bar{n}_2 &= j - 1 \\
\bar{m}_3 &= i + 1 &, \quad \bar{n}_3 &= j - 1 \\
\bar{m}_4 &= i + 1 &, \quad \bar{n}_4 &= j \\
\bar{m}_5 &= i &, \quad \bar{n}_5 &= j + 1 \\
\bar{m}_6 &= i - 1 &, \quad \bar{n}_6 &= j + 1 \\
\bar{m}_7 &= i - 1 &, \quad \bar{n}_7 &= j
\end{aligned} \tag{10}$$

For control points on the line of symmetry ($j = 0$) only three of these triangles appear in computation. Therefore, the representation within the area depends only upon five values of h (see Figure 5).

$$\begin{aligned}
\bar{m}_1 &= i &, \quad \bar{n}_1 &= 0 \\
\bar{m}_2 &= i + 1 &, \quad \bar{n}_2 &= 0 \\
\bar{m}_3 &= i &, \quad \bar{n}_3 &= 1 \\
\bar{m}_4 &= i - 1 &, \quad \bar{n}_4 &= 1 \\
\bar{m}_5 &= i - 1 &, \quad \bar{n}_5 &= 0
\end{aligned} \tag{11}$$

The function h appears in the program in the form h_{mnk} where m and n identify the corner point of the net and k refers to the time argument (τ or t). Let Δt be the time interval for which the computation is carried out. Then

$$t = k\Delta t$$

During the process of integrating over a specific subarea the corner points are characterized by one subscript (usually ℓ). We use the notation $\bar{h}_\ell$. If the $\tau/\Delta t$ is not an integer, we then

write $\bar{h}_\ell(\tau)$. If we want to use the argument τ in h_{mn}, we denote the function by $\tilde{h}_{mn}$, so that for $\tau = k\Delta t$

$$\tilde{h}_{mn}(\tau) = \tilde{h}_{mn}(k\Delta t) = h_{mnk}$$

The values of m and n pertaining, for a specific area, to the index ℓ are listed in Eqs. (8), (9), (10), and (11). Then

$$\bar{h}_\ell(\tau) = \tilde{h}_{m(\ell),n(\ell)}(\tau) = h_{mnk}$$

if

$$\tau/\Delta t = k$$

is an integer.

SECTION VII

THE INTEGRATION PROCEDURE

The approximation of h within a certain area will appear in the form

$$h(\tau,\xi,\eta) = [\phi_1(\xi,\eta),\phi_2(\xi,\eta)\ldots]A \begin{bmatrix} \bar{h}_1(\tau) \\ \bar{h}_2(\tau) \\ \vdots \end{bmatrix}$$

Here the known functions ϕ_1, ϕ_2, etc., are the elements of a row vector. A is a matrix which depends upon the geometry of the area under consideration; $\bar{h}_1$, $\bar{h}_2$, $\cdots$ the elements of a column vector, are the values of h at the corners of the area. The dimensions of A are naturally determined by the dimensions of the row and column vectors. In most cases the functions $\phi_1,\phi_2\ldots$ are very simple. The size of the vectors (which depend upon the problem in question) is small. One has, because the derivatives $h^{(2)}$ and $h^{(3)}$ are formed at constant τ,

$$h^{(2)}(\xi,\eta) = [\phi_{1,\xi},\phi_{2,\xi}\ldots]A \begin{bmatrix} \bar{h}_1(\tau) \\ \bar{h}_2(\tau) \\ \vdots \end{bmatrix} \qquad (12)$$

$$h^{(3)}(\xi,\eta) = [\phi_{1,\eta},\phi_{2,\eta}\ldots]A \begin{bmatrix} \bar{h}_1(\tau) \\ \bar{h}_2(\tau) \\ \vdots \end{bmatrix}$$

These expressions are now substituted into the integrals occurring in Eq. (6), and the integrations are carried out over the individual areas. Taking, for simplicity, only part of the expression B_9 one then has to evaluate

$$\iint f^{(2)}(\xi,\eta)[\phi_{1,\xi},\phi_{2,\xi}\cdots]A \begin{bmatrix} \bar{h}_1(\tau) \\ \bar{h}_2(\tau) \\ \vdots \end{bmatrix} d\xi d\eta$$

The integration is to be carried out for a subarea of the wing. In most cases one or several transformations of the independent variables are necessary; sometimes the functions ϕ_j are expressed in coordinates different from the $\xi\eta$ system. Further transformations are needed in order to generate a smooth integrand in an area in which the independent variables range from -1 to +1. (Triangles in the interior of the wing at a distance from control point ij and triangles in the wake are exceptions. There one uses the form B_7. The representation for $h^{(2)}$ and $h^{(3)}$ is then so simple that one can integrate directly in the $\xi\eta$ plane, using special formulae for triangular areas.) To fix the idea for the other cases let the final independent variables be p and q. Then one obtains for the above integral

$$\int_{-1}^{+1}\int_{-1}^{+1} f^{(2)}(\xi,\eta)[\phi_{1,\xi},\phi_{2,\xi}\cdots] + f^{(3)}(\xi,\eta)[\phi_{1,\eta},\phi_{2,\eta}\cdots]A \begin{bmatrix} \bar{h}_1(\tau) \\ \bar{h}_2(\tau) \\ \vdots \end{bmatrix} \frac{\partial(\xi,\eta)}{\partial(p,q)}\, dp\, dq$$

The variables ξ and η are expressed in terms of p and q. Therefore, the whole integrand appears as a function of p and q. It can be evaluated by Gaussian integrations, separately for the p and q directions. In a familiar manner one then chooses pivotal points and multiplies by the pertinent weights (say w_p and w_q). Let

$$[\psi_1,\psi_2\cdots] = \frac{\partial(\xi,\eta)}{\partial(p,q)} \{f^{(2)}(\xi,\eta)[\phi_{1,\xi},\phi_{2,\xi}\cdots] + f^{(3)}(\xi,\eta)[\phi_{1,\eta},\phi_{2,\eta}\cdots]\}\ A$$

This is a row vector of the same dimension as the column vectors $\begin{bmatrix} \bar{h}_1(\tau) \\ \bar{h}_2(\tau) \\ \vdots \end{bmatrix}$. Then the contribution of this pivotal point to B_9 is

$$\psi_1 h_1(\tau) + \psi_2 h_2(\tau) + \ldots$$

The argument τ is given by $t - \mathrm{ret}(\xi - x, \eta - y)$ where

$$\mathrm{ret} = (1 - M^2)^{-1}[M(\xi - x) + \rho]$$

$$\rho = [(\xi - x)^2 + (\eta - y)^2]^{1/2}$$

The value of ret must be evaluated for the pivotal point under consideration. In the above expression the value of ret is the same for all functions h_1, h_2, etc. Let

$$\tilde{\mathrm{ret}} = \mathrm{ret}/\Delta t$$

and $\tilde{\mathrm{ret}}^+$ and $\tilde{\mathrm{ret}}^-$ be respectively the smallest integer greater or equal to $\tilde{\mathrm{ret}}$, and the greatest integer smaller than $\tilde{\mathrm{ret}}$. Of course

$$\tilde{\mathrm{ret}}^+ - \tilde{\mathrm{ret}}^- = 1$$

Now $h_\ell(\tau)$ is obtained by linear interpolation between

$$h_\ell(k - \tilde{\mathrm{ret}}^+) \text{ and } h_\ell(k - \tilde{\mathrm{ret}}^-)$$

To be specific

$$h_\ell(\tau) = h_\ell(k - \tilde{\mathrm{ret}}^+)(\tilde{\mathrm{ret}} - \tilde{\mathrm{ret}}^-) - h_\ell(k - \tilde{\mathrm{ret}}^-)(\tilde{\mathrm{ret}} - \tilde{\mathrm{ret}}^+)$$

Thus one obtains as contribution of one pivotal point

$$\psi_1(\tilde{ret} - \tilde{ret}^-)h_1(k - \tilde{ret}^+) - \psi_1(\tilde{ret} - \tilde{ret}^+)h_1(k - \tilde{ret}^-)$$

$$+ \psi_2(\tilde{ret} - \tilde{ret}^-)h_2(k - \tilde{ret}^+) - \psi_2(\tilde{ret} - \tilde{ret}^+)h_2(k - \tilde{ret}^-)$$

$$+ \ldots$$

Now

$$h_\ell = h_{\bar{m}(\ell)\bar{n}(\ell)}$$

To obtain the form of Eq. (7) one accumulates

$$\psi_\ell(\tilde{ret} - \tilde{ret}^-) \text{ in } C_{ij,\bar{m}_\ell,\bar{n}_\ell,\tilde{ret}^+}$$

and

$$-\psi_\ell(\tilde{ret} - \tilde{ret}^+) \text{ in } C_{ij,\bar{m}_\ell,\bar{n}_\ell,\tilde{ret}^-}$$

(This is the second occasion where the expressions $\bar{m}_\ell$ are used.) The procedure is the same for triangles in the interior of the wing and not adjacent to the point ij and triangles of the wake, except that one can use special formulae for the integration within triangles.

SECTION VIII

REPRESENTATION OF h FOR DISTANT TRIANGLES AND TRANSFORMATIONS REQUIRED FOR THE INTEGRATION PROCESS

The determination of the constants $C_{(ij),(mn),k}$ requires do-loops over i,j, m, and n. It is probably preferable to choose m and n as variables in the outer do-loops, because this reduces the amount of temporary storage. For triangles not having the control point i,j as one of their corners and not adjacent to the leading or trailing edge, h within the triangle is approximated by linear interpolation between the values at the corners. The coordinates of all corner points x_{mn} and y_{mn} are initially stored in memory. Then the program proceeds in the following steps. For an inner triangle (mn1 or mn2) one extracts from memory

$$\left.\begin{aligned} \xi_\ell &= x(\bar{m}(\ell),\ \bar{n}(\ell)) \\ \eta_\ell &= y(\bar{m}(\ell),\ \bar{n}(\ell)) \end{aligned}\right\} \quad \ell = 1,2,3 \tag{13}$$

The values $\bar{m}(\ell)$ and $\bar{n}(\ell)$ are found in Eqs. (8) and (9). It is not necessary to introduce in the program $\bar{m}(\ell)$ and $\bar{n}(\ell)$ explicitly. We write

$$h_{\bar{m}(\ell),\bar{n}(\ell)}(\tau) = \bar{h}_\ell(\tau)$$

Let

$$\left.\begin{aligned} \bar{\xi}_\ell &= \xi_\ell - \xi_1 \\ \bar{\eta}_\ell &= \eta_\ell - \eta_1 \end{aligned}\right\} \quad \ell = 2,3 \tag{14}$$

The approximation for the triangle in question is written

$$h(\tau,\xi,\eta) = [1,\bar{\xi},\bar{\eta}]\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Postulating that h assumes the values h_1, h_2, and h_3 at the respective corners one obtains

$$\begin{bmatrix} \bar{h}_1(\tau) \\ \bar{h}_2(\tau) \\ \bar{h}_3(\tau) \end{bmatrix} = M \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

with

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \bar{\xi}_2 & \bar{\eta}_2 \\ 1 & \bar{\xi}_3 & \bar{\eta}_3 \end{bmatrix}$$

Hence

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = M^{-1} \begin{bmatrix} \bar{h}_1(\tau) \\ \bar{h}_2(\tau) \\ \bar{h}_3(\tau) \end{bmatrix}$$

$$h(\tau,\xi,\eta) = [1,\bar{\xi},\bar{\eta}]M^{-1}\begin{bmatrix} \bar{h}_1(\tau) \\ \bar{h}_2(\tau) \\ \bar{h}_3(\tau) \end{bmatrix}$$

In principle, at least, one sets up the matrix M and then forms M^{-1}. Actually one needs only $h^{(2)} = \partial h/\partial \xi$ and $h^{(3)} = \partial h/\partial \eta$. One has

$$h^{(2)} = [0,1,0]M^{-1} \begin{bmatrix} \bar{h}_1(\tau) \\ \bar{h}_2(\tau) \\ \bar{h}_3(\tau) \end{bmatrix}$$

$$h^{(3)} = [0,0,1]M^{-1} \begin{bmatrix} \bar{h}_1(\tau) \\ \bar{h}_2(\tau) \\ \bar{h}_3(\tau) \end{bmatrix}$$

Since the first element of the row vector on the right is zero in both cases the first row of the matrix M^{-1} is not needed. We write, accordingly

$$h^{(2)} = [1,0]A \begin{bmatrix} \bar{h}_1(\tau) \\ \bar{h}_2(\tau) \\ \bar{h}_3(\tau) \end{bmatrix}$$

$$h^{(3)} = [0,1]A \begin{bmatrix} \bar{h}_1(\tau) \\ \bar{h}_2(\tau) \\ \bar{h}_3(\tau) \end{bmatrix} \tag{15}$$

Because of the simplicity of this problem one can write down the elements of A directly (without calling a matrix inversion routine). One has

$$\mathrm{Det} = [\bar{\xi}_2 \; \bar{\eta}_3 - \bar{\xi}_3 \; \bar{\eta}_2] \tag{16}$$

The area of the triangle is

$$\mathrm{Det}/2$$

Then

$$\begin{aligned}
A_{11} &= -(A_{12} + A_{13}) \\
A_{12} &= \bar{\eta}_3/\mathrm{Det} \\
A_{13} &= -\bar{\eta}_2/\mathrm{Det} \\
A_{21} &= -(A_{22} + A_{23}) \\
A_{22} &= -\,\bar{\xi}_3/\mathrm{Det} \\
A_{22} &= \bar{\xi}_2/\mathrm{Det}
\end{aligned} \tag{17}$$

Now one applies the integration formulae for triangles reproduced in Appendix D of Reference 1 and determines the contributions to $C_{(ij),(mn),\ell}$ according to the procedure in Section VI.

Triangles adjacent to the leading edge are characterized by the indices (m = 0, n, k), k = 1,2. For them the representation of h are found in a coordinate system u,v in which the u direction is normal to the leading edge. ξ_ℓ, η_ℓ and $\bar{\xi}_\ell, \bar{\eta}_\ell$ are again defined by Eqs. (13) and (14). To transform to the u,v system we define (Figure 6)

$$\cos\alpha = \bar{\eta}_3/(\xi_3^2 + \bar{\eta}_3^2)^{1/2} \qquad \sin\alpha = \bar{\xi}_3/(\bar{\xi}_3^2 + \bar{\eta}_3^2)^{1/2} \tag{18}$$

Here α is the local sweep angle. Only $\cos\alpha$ and $\sin\alpha$, not α itself, will be needed. Now one sets

$$\begin{aligned} \bar{\xi} &= u\cos\alpha + v\sin\alpha \quad , \quad & u &= \bar{\xi}\cos\alpha - \bar{\eta}\sin\alpha \\ \bar{\eta} &= -u\sin\alpha + v\cos\alpha \quad , \quad & v &= \bar{\xi}\sin\alpha + \bar{\eta}\cos\alpha \end{aligned} \tag{19}$$

$$\frac{\partial(\xi,\eta)}{\partial(u,v)} = 1$$

At the leading edge one has $h = 0$. The steady state problem suggests that $h \sim u^{1/2}$.* One therefore has for a triangle ($m = 0,n,1$)

$$h = u^{1/2}\, u_2^{-1/2}\, \bar{h}_2(\tau) \tag{20}$$

One has

$$u_1 = u_3 = 0$$

and one computes

$$u_2 = \bar{\xi}_2 \cos\alpha - \bar{\eta}_2 \sin\alpha$$

---

*The Prandtl-Glauert distortion changes the sweep angle and the normal to the leading or trailing edges. The normal in the xy-system does not transform into the normal in the (distorted) $\bar{\xi}\eta$-system. Therefore, we have returned in the present report to the original coordinates.

To carry out the integrations needed in $\bar{B}_7$ for a triangle (m = 0,n,1) one has to form $h^{(2)}$ and $h^{(3)}$. One obtains from Eq. (19) and (20)

$$h^{(2)} = \frac{1}{2} u^{-1/2} \cos\alpha\, u_2^{-1/2}\, \bar{h}_2(\tau)$$

$$h^{(3)} = -\frac{1}{2} u^{-1/2} \sin\alpha\, u_2^{-1/2}\, \bar{h}_2(\tau)$$

(These expressions have the form of Eqs. (12), but all vectors have dimension 1.) The functions $f^{(2)}$ and $f^{(3)}$ are regular within the triangle, but $h^{(2)}$ and $h^{(3)}$ for $u = 0$ behave as $u^{-1/2}$. Moreover, one must take the triangular shape of the element into account. For the latter purpose we introduce a system of coordinates centered at the point 2 (Figure 7).

$\tilde{u} = -(u - u_2)$ $\qquad$ $u = u_2 - \tilde{u}$ $\qquad$ (rotation by π and translation)

$\tilde{v} = -(v - v_2)$ $\qquad$ $v = v_2 - \tilde{v}$

and set

$$w = \frac{\tilde{v}}{\tilde{u}}$$

Then

$$\frac{\partial(\xi\eta)}{\partial(\tilde{u},w)} = \tilde{u}\, d\tilde{u}\, dw$$

One has for the corner points

$$\tilde{u}_1 = \tilde{u}_3 = u_2$$

$$\tilde{u}_2 = 0$$

(because $u_1 = u_3 = 0$, and $u_2 = u_2$.) Moreover

$$\tilde{v}_1 = -(v_1 - v_2)$$

$$\tilde{v}_3 = -(v_3 - v_2)$$

$$\tilde{v}_2 = 0$$

The limits of integration are then

$$\tilde{u} = 0 \text{ and } \tilde{u} = u_2$$

and

$$w = w_3 = -\frac{(v_3 - v_2)}{u_2}$$

and

$$w = w_1 = -\frac{(v_1 - v_2)}{u_2} = \frac{v_2}{u_2}$$

To eliminate the singularity caused by the factor $u^{-1/2} = (u_2 - \tilde{u})^{-1/2}$, we set

$$(u_2 - \tilde{u}) = p^2$$

$$-d\tilde{u} = 2pdp$$

The limit $\tilde{u} = 0$ give $p = u_2^{1/2}$; the limit $\tilde{u} = u_2$ gives $p = 0$. The Jacobian becomes

$$\frac{\partial(\xi,\eta)}{\partial(p,w)} = -2p\tilde{u}dwdp$$

The Jacobian is negative, but this is compensated by the fact that the upper limit in p is smaller than the lower limit.

By these transformations the area integration becomes a rectangle in the wp-plane, and the integral is a smooth function because the factor p in the Jacobian cancels the denominator $u^{1/2} = (u_2 - \tilde{u})^{1/2} = p$. This integral offers no difficulties in carrying out a Gaussian integration for the p and w coordinates. One may introduce a further transformation so that the region of integration extends in both directions from -1 to +1.

$$w = (w_1 + w_3)/2 + \bar{w}(w_1 - w_3)/2)$$

$$p = (0 + u_2^{1/2})/2 + \bar{p}(0 - u_2^{1/2})/2$$

Then

$$\frac{\partial(\xi,\eta)}{\partial(\bar{p},\bar{w})} = p\ \tilde{u}((w_1 - w_3)/2)\cdot u_2^{1/2}$$

One chooses the pivotal points for the desired order of Gaussian integration in the $\bar{p}$ and $\bar{w}$ integration, and determines for the pivotal points in turn, p, w, $\tilde{u}$, $\tilde{v}$, u, v, and ξ,η. Then one evaluates the integrand (including the Jacobian) and the retardation for the pivotal points (and in the case of Gaussian integration multiplies by the weights). With these integrals one

determines the contributions to $C_{(ij),(mn),k}$ according to the procedure of Section VII.

For the triangle (m = 0,n,2) one introduces again the quantities defined in Eqs. (13) and (14). The sweep angle α is again determined by the above formulae, but one might also introduce auxiliary points 4 and 5 (Figure 8) and introduce

$$\cos\alpha = (\eta_5 - \eta_4)/[(\xi_5 - \xi_4)^2 + (\eta_5 - \eta_4)^2]^{1/2}$$

$$\sin\alpha = (\xi_5 - \xi_4)/[(\xi_5 - \xi_4)^2 + (\eta_5 - \eta_4)^2]^{1/2} \qquad (21)$$

For a straight leading edge, there is, of course, only one angle α. Now one sets for a triangle (m = 0,n,2).

$$h(\tau,\xi,\eta) = [u^{1/2}, v]\begin{bmatrix} a \\ b \end{bmatrix}$$

Then by matching h at the points 2 and 3

$$M\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \bar{h}_2(\tau) \\ \bar{h}_3(\tau) \end{bmatrix}$$

with

$$M = \begin{bmatrix} u_2^{1/2} & v_2 \\ u_3^{1/2} & v_3 \end{bmatrix}$$

Let

$$A = M^{-1}$$

Then

$$h(\xi,\eta,\tau) = [u^{1/2},v]A\begin{bmatrix}\bar{h}_2(\tau)\\ \bar{h}_3(t)\end{bmatrix}$$

Hence

$$h^{(2)} = \left[(1/2)u^{-1/2}\cos\alpha,\ v\sin\alpha\right]A\begin{bmatrix}\bar{h}_2(\tau)\\ \bar{h}_3(\tau)\end{bmatrix}$$

(22)

$$h^{(3)} = \left[-(1/2)u^{-1/2}\sin\alpha,\ v\cos\alpha\right]A\begin{bmatrix}\bar{h}_2(\tau)\\ \bar{h}_3(\tau)\end{bmatrix}$$

To take the triangular shape of the element into account a new system of coordinates is introduced, so that the side 2,3 of the triangle is a line $\tilde{u}$ = constant. For this purpose define

$$\cos\tilde{\alpha} = (v_3 - v_2)/[(u_3 - u_2)^2 + (v_3 - v_2)^2]^{1/2}$$

$$\sin\tilde{\alpha} = (u_3 - u_2)/[(u_3 - u_2)^2 + (v_3 - v_2)^2]^{1/2}$$

set

$$u = \tilde{u}\cos\tilde{\alpha} + \tilde{v}\sin\tilde{\alpha} \quad , \quad \tilde{u} = u\cos\tilde{\alpha} - v\sin\tilde{\alpha}$$

$$v = -\tilde{u}\sin\tilde{\alpha} + \tilde{v}\cos\tilde{\alpha} \quad , \quad \tilde{v} = u\sin\alpha + v\cos\tilde{\alpha}$$

and compute

$$u_2, v_2 \quad , \quad u_3, v_3 \quad , \quad \tilde{u}_2 = \tilde{u}_3 \quad , \text{ and } \tilde{v}_2$$

Furthermore, introduce

$$w = \tilde{v}/\tilde{u}$$

and compute w_2 and w_3. Then

$$\frac{\partial(\xi, \eta)}{\partial(\tilde{u}, w)} = \tilde{u} \, d\tilde{u} \, dw$$

Then

$$h^{(2)} = \left[\frac{1}{2} \tilde{u}^{-1/2} (\cos \tilde{\alpha} + w \sin \tilde{\alpha})^{-1/2} \cos \alpha, \ \sin \alpha\right] A \begin{bmatrix} \bar{h}_2(\tau) \\ \bar{h}_3(\tau) \end{bmatrix}$$

$$h^{(3)} = \left[-\frac{1}{2} \tilde{u}^{-1/2} (\cos \tilde{\alpha} + w \sin \tilde{\alpha})^{-1/2} \sin \alpha, \ \cos \alpha\right] A \begin{bmatrix} \bar{h}_2(\tau) \\ \bar{h}_3(\tau) \end{bmatrix}$$

Actually, one simply evaluates u and v for the pivotal points and substitutes in Eqs. (22), but the last equation shows the singularity that arises for $\tilde{u} = 0$.

The limits of integration are now

$$\tilde{u} = 0 \quad \text{and} \quad \tilde{u} = \tilde{u}_2$$

and

$$w = w_2 \text{ and } w = w_3$$

To eliminate the singularity caused by the factor $\tilde{u}^{-1/2}$ we set

$$\tilde{u} = p^2$$

Then

$$d\tilde{u} = 2pdp$$

$$\frac{\partial(\xi,\eta)}{\partial(\bar{p},w)} = 2p\tilde{u}$$

The integration is carried out over the rectangle $p = 0$ to $p = \tilde{u}_2^{1/2}$, $w = w_2$ to w_3. To make the limits of integration in both cases -1 and +1 we set

$$p = (\tilde{u}_2^{1/2}/2) + \bar{p}(\tilde{u}_2^{1/2}/2)$$

$$w = ((w_2 + w_3)/2) + \bar{w}((w_3 - w_2)/2))$$

Then

$$\frac{\partial(\xi,\eta)}{\partial(\bar{p},\bar{w})} = \frac{pu}{2}\,\tilde{u}_2^{1/2}(w_3 - w_2)$$

From here on the procedure is carried out in the same manner as above.

The two triangles at the trailing edge ($m = i_t-1,n,1$) and ($m = i_t-1,n,2$) are considered at the same time (Figure 9). To determine h one needs an additional point in the wake (point 5). It is assumed to lie in the wake of point 2. One has

$$\bar{h}_5(\tau) = \bar{h}_2(\tau - \frac{1}{M}(\xi_5 - \xi_2))$$

One has here

$$\bar{m}_1 = i_{t-1} \quad , \quad \bar{n}_1 = n$$

$$\bar{m}_2 = i_t \quad , \quad \bar{n}_2 = n$$

$$\bar{m}_3 = i_{t-1} \quad , \quad \bar{n}_3 = n + 1$$

$$\bar{m}_4 = i_t \quad , \quad \bar{n}_4 = n + 1$$

$$\bar{m}_5 = i_{t+1} \quad , \quad \bar{n}_5 = n$$

One determines ξ_ℓ and η_ℓ ($\ell = 1...5$)

and

$$\begin{aligned} \bar{\xi}_\ell &= \xi_\ell - \xi_2 \\ \bar{\eta}_\ell &= \eta_\ell - \eta_2 \end{aligned} \qquad \ell = 1...5$$

Furthermore, one evaluates for the sweep angle.

$$\cos\alpha = \bar{\eta}_4 / (\bar{\xi}_4^2 + \bar{\eta}_4^2)^{1/2} \quad , \quad \sin\alpha = \bar{\xi}_4 / (\bar{\xi}_4^2 + \bar{\eta}_4^2)^{1/2}$$

carries out the rotation of the coordinate system

$$u = \bar{\xi}\cos\alpha - \bar{\eta}\sin\alpha \quad ; \quad \bar{\xi} = u\cos\alpha + v\sin\alpha$$

$$v = \bar{\xi}\sin\alpha + \bar{\eta}\cos\alpha \quad ; \quad \bar{\eta} = -u\sin\alpha + v\cos\alpha$$

and determines the values of $\bar{u}$ and $\bar{v}$ for points 1 through 5. In the wake there is no singularity of h at the trailing edge.

$$h = [1,u,v]\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

The expression for h at the wing differs from that for the wake by a term of the form $(-u)^{3/2}$. Then h and its first derivatives are continuous at the trailing edge. Therefore on the wing

$$h = [1, u, v, (-u)^{3/2}] \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

To determine h in the triangle $(m = i_t - 1, n, 2)$, it is matched at the points 2, 3, 4, and in addition at point 5. This is expressed by

$$\begin{bmatrix} h_2 \\ h_4 \\ h_5 \\ h_3 \end{bmatrix} = M \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

with

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & v_4 & 0 \\ 1 & u_5 & v_5 & 0 \\ 1 & u_3 & v_3 & (-u_3)^{3/2} \end{bmatrix}$$

Compute

$$A = M^{-1}$$

Then

$$h = [1,u,v,0]A\begin{bmatrix} h_2 \\ h_4 \\ h_5 \\ h_3 \end{bmatrix}$$

in the wake,

and

$$h = [1,u,v,(-u)^{3/2}]A\begin{bmatrix} h_2 \\ h_4 \\ h_5 \\ h_3 \end{bmatrix}$$

on the wing. The representation for the wake can, of course, be obtained more directly, but the present form is somewhat easier to program.

For the triangle ($m = i_{t-1}, n, 1$) one matches at the points 1, 2, 3, and 5. Then

$$h = [1,u,v,(-u)^{3/2}]\begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_5 \end{bmatrix}$$

and the pertinent matrix M is given by

$$M = \begin{bmatrix} 1 & u_1 & v_1 & (-u_1)^{3/2} \\ 1 & 0 & 0 & 0 \\ 1 & u_3 & v_3 & (-u_3)^{3/2} \\ 1 & u_2 & v_2 & 0 \end{bmatrix}$$

Compute

$$A = M^{-1}$$

Then

$$h = [1, u, v, (-u)^{3/2}]A \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_5 \end{bmatrix}$$

Now one has for the triangle $(m = i_{t-1}, n, 1)$

$$h^{(2)} = \left\{ \cos\alpha[0,1,0,-\tfrac{3}{2}(-u)^{1/2}] + \sin\alpha[0,0,1,0] \right\} A \begin{bmatrix} h_1(\tau) \\ h_2(\tau) \\ h_3(\tau) \\ h_5(\tau) \end{bmatrix}$$

$$h^{(3)} = \left\{ \sin\alpha[0,-1,0,\ \tfrac{3}{2}(-u)^{1/2}] + \cos\alpha[0,0,1,0] \right\} A \begin{bmatrix} h_1(\tau) \\ h_2(\tau) \\ h_3(\tau) \\ h_5(\tau) \end{bmatrix}$$

and for the triangle $(m = i_{t-1}, n, 2)$

$$h^{(2)} = \left\{\cos \alpha[0,1,0,-(3/2)(-u)^{1/2}] + \sin \alpha[0,0,1,0]\right\} A \begin{bmatrix} h_2(\tau) \\ h_4(\tau) \\ h_5(\tau) \\ h_3(\tau) \end{bmatrix}$$

$$h^{(3)} = \left\{\sin \alpha[0,-1,0,(3/2)(-u)^{1/2}] + \cos \alpha[0,0,1,0]\right\} A \begin{bmatrix} h_2(\tau) \\ h_4(\tau) \\ h_5(\tau) \\ h_3(\tau) \end{bmatrix}$$

(Of course with different matrices A.)

The further steps of the treatment are different for the triangles $(m = i_t-1,n,1)$ and $(m = i_t-1,n,2)$. The side (1-3) of the triangle $(m = i_t-1,n,1)$ is not necessarily parallel to the trailing edge, therefore, a rotation of the coordinate system is carried out so that $\tilde{u}_1$ equals $\tilde{u}_3$ (Figure 10).

$$\cos \tilde{\alpha} = (v_1 - v_3)/[(u_1 - u_3)^2 + (v_1 - v_3)^2]^{1/2}$$

$$\sin \tilde{\alpha} = (u_1 - u_3)/[(u_1 - u_3)^2 + (v_1 - v_3)^2]^{1/2}$$

$$u = \tilde{u} \cos \tilde{\alpha} + \tilde{v} \sin \tilde{\alpha} \quad , \quad \tilde{u} = u \cos \tilde{\alpha} - v \sin \tilde{\alpha}$$

$$v = -\tilde{u} \sin \tilde{\alpha} + \tilde{v} \cos \tilde{\alpha} \quad , \quad \tilde{v} = u \sin \tilde{\alpha} + v \cos \tilde{\alpha}$$

Set

$$w = \tilde{v}/\tilde{u}$$

Then

$$\frac{\partial(\xi,\eta)}{\partial(\tilde{u},w)} = \tilde{u}\, d\tilde{u}\, dw$$

The limits of integration are 0 and $\tilde{u}_3$ for $\tilde{u}$ and w_3 and w_1 for w. The factor $(-u)^{1/2}$ gives rise to a term

$$(\tilde{u})^{1/2}(-\cos \alpha + w \sin \alpha)^{1/2}$$

Therefore we introduce

$$\tilde{u} = p^2$$

Then

$$\frac{\partial(\xi, \eta)}{\partial(p, w)} = 2\tilde{u}p$$

Again we transform p and w so that in either case the limits are -1 and +1.

$$p = ((\tilde{u}_3{}^{1/2})/2) + \bar{p}((\tilde{u}_3{}^{1/2})/2)$$

$$w = (w_3 + w_1)/2 + \bar{w}(w_1 - w_3)/2$$

Then

$$\frac{\partial(\xi, \eta)}{\partial(\bar{p}, \bar{w})} = (1/2)\tilde{u}p(\tilde{u}_3)^{1/2}(w_1 - w_3)$$

The argument of h_5 needs special attention. One has

$$h_5(\tau) = h_2(\tau - \frac{1}{M}(\xi_5 - \xi_2))$$

with

$$\tau = t - ret$$

Therefore, one forms

$$\hat{ret} = [ret + \frac{1}{M}(\xi_5 - \xi_2)]/\Delta t$$

determines the values of ξ from $\hat{ret}^+$ and $\hat{ret}^-$, and accumulates these contributions in $\bar{h}_2$ for $\xi = \hat{ret}^+$ and $\hat{ret}^-$. Actually, this is the procedure for all points in the wake.

For the triangle ($m = i_{t-1}, n, 2$) one first shifts the coordinate system to point 3 (Figure 11).

$$\bar{u} = u - u_3 \quad , \quad u = u_3 + \bar{u}$$

$$\bar{v} = v - v_3 \quad , \quad v = v_3 + \bar{v}$$

$$w = \bar{v}/\bar{u}$$

The term $(-u)^{1/2}$ becomes $(-u_3 + \bar{u})^{1/2}$. To counteract the singularity caused by this term we set

$$p^2 = (-u_3 + \bar{u})$$

Then

$$2p\ dp = -d\bar{u}$$

The limits for p are $(-u_3)^{1/2}$ and 0, the limits for w are w_5 and w_4. The Jacobian is

$$\frac{\partial(\xi,\eta)}{\partial(p,w)} = -2p\bar{u}\ dp\ dw$$

So far one has the integration over a rectangle. To reduce this to the integration over a square we set

$$p = ((-u_3)^{1/2}/2) - \bar{p}(-u_3)^{1/2}/2$$

$$w = (w_2 + w_4)/2 + \bar{w}(w_4 - w_2)/2$$

Then

$$\frac{\partial\ (\xi\eta)}{\partial(\bar{p},\bar{w})} = (1/2)p\bar{u}(-u_3)^{1/2}(w_4 - w_2)$$

SECTION IX

APPROXIMATION FOR h IN THE AREA SURROUNDING A CONTROL POINT AND TRANSFORMATIONS REQUIRED FOR THE INTEGRATION

The right-hand side of the integral equation (that is the expression B_9) is particularly sensitive to the approximation chosen for the function h. In the computation only values of h at the corner points of the grid are available. A local mesh refinement cannot be carried out, because eventually each corner point will be a control point; ultimately one would have an overall refinement of the grid.

The fact that $h^{(2)}(x_i, y_i) = \frac{\partial h}{\partial x}(x_i, y_i)$ and $h^{(3)}(x_i, y_i) = \frac{\partial h}{\partial y}(x_i, y_i)$ occur in a number of expressions contributing to $\partial h(x_i, y_i)/\partial t$ may prove a source of uncertainty. The contributions of some of these terms do not go to zero as one considers a smaller and smaller neighborhood of the control point (which shows that these terms are indeed important). There may be a partial cancellation of these terms. Such cancellations would result in a loss of accuracy, if one approximates in different triangles adjacent to the control point the functions $h^{(2)}$ and $h^{(3)}$ by different expressions. It is, therefore, preferable to use one analytic expression for h throughout the entire area surrounding the control point under consideration. Actually, one cannot guarantee that h is an analytic function throughout the area, and one cannot claim that such a representation has higher accuracy that an approximation by piecewise linear functions. The higher order approximation is solely used to avoid an inconsistency. In the realization of the method which we have in mind, the function $w_0(x,y)$ has the form of some function in x and y multiplied by a step function in time (details in Section XI). The potential then approaches a steady state. The steady state is given by a smooth function provided, of course, that the dependence of w_0 upon x and y is smooth. The approximation of h in the vicinity of a control point by a polynomial is then advantageous because of its greater accuracy.

In studying the question of accuracy in detail, one can study a fictitious problem where, in the vicinity of a control point, h is exactly given by second degree polynomial. (One evaluates the exact contribution to the upwash, once using the exact formula for h, a second time with a linear approximation for each of the six triangles and compares the results.)

The choice of the approximation for h in the area surrounding a control point requires some discussion. A polynomial of the second degree contains six arbitrary constants but one must match h at seven points (six corners of the hexagon and the center). A third degree polynomial, on the other hand, contains too many arbitrary constants.

The situation can be discussed in detail for a regular hexagon, and nearly all the results can be carried over to a hexagon which arises by an affine transformation.

To see the question in a more general light we carry out the initial discussion for a regular octagon. Assume initially that the corners lie on a circle with radius 1, that the center lies at the origin of the x,y-system, and that two corners lie on the x-axis. First we disregard the matching at the origin. Let $\mathrm{arctg}(y/x) = \theta$. The matching is then carried out at equidistant values of θ, namely $\theta = \frac{n2\pi}{8}$, $n = 1 \ldots 8$. It can be accomplished by a discrete Fourier analysis. The functions involved are 1, $\cos\theta$, $\sin\theta$, $\cos 2\theta$, $\sin 2\theta$, $\cos 3\theta$, $\sin 3\theta$, and $\cos 4\theta$. The function $\sin 4\theta$ does not appear because it is zero at all matching points.

The case in which no corners lie on the x-axis can be studied by rotating the coordinate system by, say, an angle ϵ. The term $\cos 4\theta$ then appears as

$$\cos(4(\theta - \epsilon)) = \cos(4\epsilon)\cos(4\theta) + \sin(4\epsilon)\sin(4\theta)$$

and one must admit $\sin(4\theta)$ as well as $\cos(4\theta)$. The excluded function is then

$$\sin(4(\theta - \varepsilon)) = \cos(4\varepsilon)\sin(4\theta) - \sin(4\varepsilon)\cos(4\theta)$$

A formulation in which ε need not be specified is obtained if one admits cos (4θ) as well as sin (4θ) and postulates that the sum of the square of their coefficients be minimized.

If one omits in this problem $\cos(4\theta)$ and $\sin(4\theta)$ and postulates that the sum of the square of the errors be minimized, then it follows from the properties of a finite Fourier decomposition that all coefficients up to those of $\cos(3\theta)$ and $\sin(4\theta)$ remain unchanged. The errors are those caused by the omitted term $\cos(4\theta)$ or $\cos(4(\theta - \varepsilon))$. They have the form ± 1 multiplied by a constant, where adjacent points have opposite signs. This is a reasonable approximation.

Actually, one carries out the approximation in terms of a polynomial in x and y; in the case under consideration this is a fourth degree polynomial. The constant in the polynomial is the only term which affects the value at the center of the octagon (the origin of the x,y-system). Postulating a perfect match at the center, one has to match the values of $h_{corner} - h_{center}$ at the corners. For this purpose one has the following expressions at one's disposal

$$x,\ y,\ x^2,\ xy,\ y^2,\ x^3,\ x^2y,\ xy^2,\ y^3,\ x^4,\ x^3y,\ x^2y,\ xy^3,\ \text{and}\ y^4$$

The expressions $\cos(n\theta)$ and $\sin(n\theta)$ of the matching by a Fourier decomposition can also be written (along the circle of radius one) as polynomials in x and y. They are

$$1,\ x,\ y,\ x^2 - y^2,\ 2xy,\ x^3 - 3xy^2,\ 3x^2y - y^3,\ x^4 - 6x^2y^2 + y^3$$

and, as a counterpart to sin 4θ, $4x^3y - 4xy^3$.

The following observations are pertinent for a comparison of these two lists of terms. For all corners of the octagon

error the contribution of the omitted third degree terms; they are of the form ± 1 multiplied by a constant. For this omitted term the deviation of h in the interior of the hexagon behave as r^3 (with $r^2 = x^2 + y^2$). Accordingly, deviations are rather small at some distance from the outer edges. To allow for matching at the center, one also admits the constant 1.

We have assumed in the derivation that one has a perfect match at the center. For the regular hexagon this does not affect the minimization process; if $h = 1$ in the middle and zero at all corner points, then the expression $1 - x^2 - y^2$ gives a perfect matching. The requirements for perfect matching by third degree polynomials have been stated above. It can be assumed that for hexagons that do not deviate too strongly from regular hexagons the conditions developed here are satisfactory.

The numbering of the points [illegible] hexagon including the control point itself goes from [illegible] 4. For every control point one [illegible] the seven points where h is given.

[illegible]

where $\bar{m}(\ell)$ and $\bar{n}(\ell)$ are defined [illegible] control point in the interior. We int[illegible] centered at the control point

$$\bar{\xi} = \xi - \xi_1 \qquad \qquad \qquad (23)$$
$$\bar{\eta} = \eta - \eta_1$$

If one expresses h by a quadratic function, then one has six coefficients at one's disposal. The approximation for a quadratic expression is written in the form

$x^2 + y^2 = 1$: In the first list the term 1 does not appear, therefore, the terms x^2, xy, y^2 of the first list express, as far as the matching is concerned the same family of functions as 1, $x^2 - y^2$ and $2xy$. In the first list some of the expressions of the third and fourth order can be expressed by terms already included in the list because $x^2 + y^2 = 1$. They are

$$xy^2 = x - x^3,\ y^3 = y - yx^2,\ x^2y^2 = x^2 - x^4,\ xy^3 = -xy - x^3y$$

Accordingly, they ought to be omitted. Besides one has the expression $\sin 4\theta = 4x^3y - 4xy^3$ which vanishes. Accordingly, the first list contains a number of terms which are linearly dependent. The matching by a fourth order polynomial in x and y is therefore best carried out with terms of the second list, where the first term 1 is expressed as $x^2 + y^2$. Accordingly, one has

$$x,\ y,\ x^2,\ xy,\ y^2,\ x^3 - 3xy^2,\ 3x^2y - y^3,\ x^4 - 6x^2y^2 + y^4$$

and $4x^3y - 4xy^2$

with the postulate that the sum of the squares of the coefficients of the last two terms be minimized for the regular octagon. This may appear nearly trivial for the regular octagon but, by using the polynominal expressions, the procedure can be carried over to general octagons. Of course, the nature of the approximation is then only loosely defined.

The case of an octagon has been chosen because it shows clearly how the fact that $x^2 + y^2 = 1$ at the corners, eliminates some of the expressions of a general polynomial (because they are linearly dependent upon expressions already included). In the case of a hexagon a polynomial of the third degree is sufficient. One then admits the terms x, y, x^2, xy, y^2, $x^3 - 3xy^2$, and $3x^2 - y^3$ (after one has matched at the center), and minimizes the sum of the squares of the coefficients of the last two expressions. If one omits the terms of the third degree and minimizes the sum of the squares of the errors, one obtains as

$$h(\xi,\eta,\bar{k}) = (1,\bar{\xi},\bar{\eta},\bar{\xi}^2,\bar{\xi}\bar{\eta},\bar{\eta}^2)\begin{bmatrix} a_1(\bar{k}) \\ \vdots \\ a_7(\bar{k}) \end{bmatrix}$$

(Here $\bar{\xi}$ and $\bar{\eta}$ on the right are considered as functions of ξ and η (Eq. 23).)

Matching at points 1 to 7 would lead to the requirement that

$$\begin{bmatrix} M & \begin{matrix} a_1 \\ \vdots \\ a_6 \end{matrix} \end{bmatrix} \text{ be equal to } \begin{bmatrix} \bar{h}_1(\tau) \\ \vdots \\ \bar{h}_7(\tau) \end{bmatrix}$$

where

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & \xi_2 & \eta_2 & \xi_2^2 & \xi_2\eta_2 & \eta_2^2 \\ 1 & \xi_3 & \eta_3 & \xi_3^2 & \xi_3\eta_3 & \eta_3^2 \\ 1 & \xi_4 & \eta_4 & \xi_4^2 & \xi_4\eta_4 & \eta_4^2 \\ 1 & \xi_5 & \eta_5 & \xi_5^2 & \xi_5\eta_5 & \eta_5^2 \\ 1 & \xi_6 & \eta_6 & \xi_6^2 & \xi_6\eta_6 & \eta_6^2 \\ 1 & \xi_7 & \eta_7 & \xi_7^2 & \xi_7\eta_7 & \eta_7^2 \end{bmatrix}$$

Written in detail this would require that

$$\sum_{\ell=1}^{6} M_{i\ell}a_\ell - h_i \quad ; \quad i = 1\ldots7$$

be zero. In general this is impossible. In accordance with the discussion at the beginning of this section we impose the weaker requirement

$$\sum_{i=1}^{7} \sum_{\ell=1}^{6} [M_{i\ell}a_\ell - h_i]^2 = \text{Min} \tag{24}$$

Hence, by differentiation with respect to a_m

$$\sum_{i=1}^{7} M_{im} \sum_{\ell=1}^{6} (M_{i\ell} a_\ell - h_i) = 0 \quad ; \quad m = 1 \ldots 6$$

This is written in terms of matrices

$$M^+ M \begin{bmatrix} a_1 \\ \vdots \\ a_6 \end{bmatrix} - M^+ \begin{bmatrix} h_1 \\ \vdots \\ h_7 \end{bmatrix} = 0$$

Hence

$$\begin{bmatrix} a_1 \\ \vdots \\ a_6 \end{bmatrix} = A \begin{bmatrix} h_1 \\ \vdots \\ h_7 \end{bmatrix}$$

with

$$A = [M^+ M]^{-1} M^+$$

This procedure is known as forming the generalized inverse of the matrix M. In the computation one first builds up the matrix M and subsequently forms M^+M, $[M^+M]^{-1}$, and

$$A = [M^+ M]^{-1} M^+$$

The representation for h is then given by

$$h(\xi, \eta, \bar{k}) = [1, \bar{\xi}, \bar{\eta}, \bar{\xi}^2, \bar{\xi}\bar{\eta}, \bar{\eta}^2] A \begin{bmatrix} h_1 \\ \vdots \\ h_7 \end{bmatrix} \qquad (25)$$

Perfect matching at all seven points is achieved by admitting expressions of the third degree, but according to the above discussion the third degree terms are chosen in a special form.

As we postulate perfect matching at all points, we introduce

$$\bar{h}(\xi,\eta) = h(\xi,\eta) - h(\xi_1,\eta_1),$$

that is, we anticipate the matching at the pivotal point 1. Then we have a representation

$$\bar{h}(\bar{\xi},\bar{\eta}) = [\bar{\xi},\bar{\eta},\bar{\xi}^2,\bar{\xi}\bar{\eta},\bar{\eta}^2,(\bar{\xi}^3 - 3\bar{\xi}\bar{\eta}^2),(3\bar{\xi}^2\bar{\eta} - \bar{\eta}^3)]\begin{bmatrix} a_1 \\ \vdots \\ a_7 \end{bmatrix}$$

Notice the special form of the terms of third degree. We partition the vector $[\leftarrow a_i \rightarrow]$

$$[a_1 \ldots a_7] = [u_1 \ldots u_5 \vdots v_1, v_2]$$

This means that we denote the coefficients of the linear and quadratic terms by $u_1 \ldots u_5$ and the coefficients of the two third degree terms by v_1 and v_2. Then

$$[\bar{\xi},\bar{\eta},\bar{\xi}^2,\bar{\xi}\bar{\eta},\bar{\eta}^2]\begin{bmatrix} u_1 \\ \vdots \\ u_5 \end{bmatrix} + [\bar{\xi}^3 - 3\bar{\xi}\bar{\eta}^2, 3\bar{\xi}^2\bar{\eta} - \bar{\eta}^3]\begin{bmatrix} v_1 \\ \\ v_2 \end{bmatrix} = \bar{h}(\bar{\xi},\bar{\eta})$$

We choose arbitrarily five corners of the hexagon. For instance, the corners 2 to 6. Next, define a 5 by 5 matrix B by

$$\left.\begin{aligned} B_{i1} &= \bar{\xi}_{i+1} \\ B_{i2} &= \bar{\eta}_{i+1} \\ B_{i3} &= \bar{\xi}^2_{i+1} \\ B_{i4} &= \bar{\xi}_{i+1}\,\bar{\eta}_{i+1} \\ B_{i5} &= \bar{\eta}^2_{i+1} \end{aligned}\right\} \quad i = 1 \ldots 5$$

a 5 by 2 matrix C by

$$C_{i1} = \bar{\xi}_{i+1}^{\,3} - 3\bar{\xi}_{i+1}\,\bar{\eta}_{i+1}^{\,2} \qquad\qquad i = 1 \ldots 5$$

$$C_{i2} = 3\bar{\xi}_{i+1}^{\,2}\,\bar{\eta}_{i+1} - \bar{\eta}_{i+1}^{\,3}$$

a row vector c with 5 elements considered as a 1 by 5 matrix by

$$c_1 = \bar{\xi}_7, \; c_2 = \bar{\eta}_7, \; c_3 = \bar{\xi}_7^2, \; c_4 = \bar{\xi}_7\bar{\eta}_7, \; c_5 = \bar{\eta}_7^2$$

and another row vector d with two elements

$$d_1 = \bar{\xi}_7^3 - 3\bar{\xi}_7\bar{\eta}_7^2 \text{ and } d_2 = 3\bar{\xi}_7\bar{\eta}_7^2 - \bar{\eta}_7^3$$

Finally, we introduce the column vectors $\hat{h}$ with 6 elements, and $\tilde{h}$, which is the vector $\hat{h}$ truncated after the fifth element.

$$\hat{h}_i = \bar{h}(\bar{\xi}_{i+1}, \bar{\eta}_{i+1}) \quad , \quad i = 1 \ldots 6$$

$$\tilde{h}_i = \bar{h}(\bar{\xi}_{i+1}, \bar{\eta}_{i+1}) \quad , \quad i = 1 \ldots 5$$

The matching conditions at the points 2 through 6 and at point 7 are now written separately.

$$Bu + Cv = \tilde{h} \tag{26}$$

$$cu + dv = \hat{h}_7 \tag{27}$$

One obtains from Eq. (26)

$$u = B^{-1}(\tilde{h} - Cv) \tag{28}$$

Substituting this into Eq. (27) and collecting terms with v one obtains

$$ev + f\hat{h} = 0 \tag{29}$$

with

$$e = d - cB^{-1}C \tag{30}$$

$$f = [f_1, \ldots, f_6] = [cB^{-1} \vdots - 1] \tag{31}$$

(e and f are row vectors with 2 and 6 elements, respectively.) Eq. (29) is one equation for the two unknowns v_1 and v_2. Written explicitly it reads

$$e_1 v_1 + e_2 v_2 + \sum_{j=1}^{6} f_j \hat{h}_j = 0 \tag{32}$$

Eq. (32) is obtained by postulating that the function $\bar{h}(\xi, \eta)$ is matched at points 2 to 7. As Eq. (32) is one equation with two unknowns, the result is not uniquely determined. We now introduce the postulate that the sum of the squares of the coefficients of the third degree terms be a minimum.

$$(1/2)(v_1{}^2 + v_2{}^2) = \min$$

Eq. (32) is a constraint imposed on this minimization. After introducing a Lagrange multiplier λ, one must determine the values of v_1 and v_2 for which the expression

$$(1/2)(v_1{}^2 + v_2{}^2) + \lambda[e_1 v_1 + e_2 v_2 + \sum_{j=1}^{6} f_j \hat{h}_j]$$

is stationary. By differentiation with respect to v_1 and v_2 one obtains

$$v_i + \lambda e_i = 0 \quad , \quad i = 1,2$$

and by substitution into the constraint Eq. (32)

$$\lambda = f\hat{h}(e_1{}^2 + e_2{}^2)^{-1}$$

Hence

$$v = -(e_1^2 + e_2^2)^{-1} e^{+} f \hat{h}$$

(One remembers that v is a column vector and e a row vector.) This is rewritten in the form

$$v = M_2 \hat{h}$$

with

$$M_2 = -(e_1^2 + e_2^2)^{-1} e^{+} f$$

Then from Eq. (28)

$$u = M_1 \hat{h}$$

with

$$M_1 = B^{-1} \left\{ \begin{bmatrix} & \cdot & 0 \\ & \cdot & 0 \\ I_5 & \cdot & 0 \\ & \cdot & 0 \\ & \cdot & 0 \end{bmatrix} - CM_2 \right\}$$

Here I_5 is the five by five identity matrix. Then

$$\begin{bmatrix} a_1 \\ \vdots \\ a_7 \end{bmatrix} = \begin{bmatrix} u \\ \cdots \\ v \end{bmatrix} = M_3 \hat{h}$$

with

$$M_3 = \begin{bmatrix} M_1 \\ \cdots \\ M_2 \end{bmatrix}$$

In the program one generates the matrices B and C and the vectors c and d and computes

$$f = [cB^{-1} \; - 1]$$

$$e = d - cB^{-1}C$$

$$M_2 = -(e_1{}^2 + e_2{}^2)^{-1}e^{+}f$$

$$M_1 = B^{-1}\left\{\begin{bmatrix} & \vdots & 0 \\ & \vdots & 0 \\ I_5 & \vdots & 0 \\ & \vdots & 0 \\ & \vdots & 0 \end{bmatrix} - CM_2\right\}$$

$$A_2 = \begin{bmatrix} M_1 \\ \cdots \\ M_2 \end{bmatrix}$$

Then one has

$$\bar{h}(\bar{\xi},\bar{\eta}) = [\bar{\xi},\bar{\eta},\bar{\xi}^2,\bar{\xi}\bar{\eta},\bar{\eta}^2,(\bar{\xi}^3 - 3\bar{\xi}\bar{\eta}^2),(\bar{\xi}^2\bar{\eta} - 3\bar{\eta}^3)]A\begin{bmatrix} \bar{h}_2 \\ \vdots \\ \bar{h}_7 \end{bmatrix}$$

The integration procedure is now described for the approximation to h discussed first, (Eq. 25). One has for a control point in the interior

$$h^{(2)} = [0,1,0,2\bar{\xi},\bar{\eta},0]A\begin{bmatrix} h_1(\tau) \\ \vdots \\ h_7(\tau) \end{bmatrix}$$

$$h^{(3)} = [0,0,1,0,\bar{\xi},2\bar{\eta}]A\begin{bmatrix} h_1(\tau) \\ \vdots \\ h_7(\tau) \end{bmatrix}$$

The expressions to be evaluated for the area adjacent to the control point ij are $\bar{B}_8$ and $\bar{B}_9$. $\bar{B}_9$ is an area integral. The integrals are evaluated separately for the six triangles containing the point (ij).

Consider a triangle with corners 1, ℓ, $\ell+1$. For $\ell = 7$ we identify the point $\ell+1$ with the point 2. We introduce a system (u,v) in which the side $(\ell, \ell+1)$ of the triangle is a line u = const. For this purpose we set

$$\cos\alpha = (\bar{\eta}_{\ell+1} - \bar{\eta}_{\ell})/[(\bar{\xi}_{\ell+1} - \bar{\xi}_{\ell})^2 + (\bar{\eta}_{\ell+1} - \bar{\eta}_{\ell})^2]^{1/2}$$

$$\sin\alpha = (\bar{\xi}_{\ell+1} - \bar{\xi}_{\ell})/[(\bar{\xi}_{\ell+1} - \bar{\xi}_{\ell})^2 + (\bar{\eta}_{\ell+1} - \bar{\eta}_{\ell})^2]^{1/2}$$

$$\begin{aligned} \bar{\xi} &= u\cos\alpha + v\sin\alpha \quad , \quad u = \bar{\xi}\cos\alpha - \bar{\eta}\sin\alpha \\ \bar{\eta} &= -u\sin\alpha + v\cos\alpha \quad , \quad v = \bar{\xi}\sin\alpha - \bar{\eta}\cos\alpha \end{aligned} \tag{33}$$

Furthermore,

$$w = v/u \quad , \quad v = uw$$

Then

$$\frac{\partial(\xi,\eta)}{\partial(u,w)} = u$$

One now has an integration over a rectangular area

$$0 < u < u_{\ell} \quad , \quad w_{\ell} < w \; w_{\ell+1}$$

To obtain in both directions limits -1 and +1 we set

$$u = u_{\ell}/2 + \bar{w}u_{\ell}/2 \quad , \quad w = (w_{\ell+1} + w_{\ell})/2 + \bar{w}(w_{\ell+1} - w_{\ell})/2$$

Then

$$\frac{\partial(\xi,\eta)}{\partial(\bar{u},\bar{w})} = \frac{1}{4}\, u\, u_{\ell}(w_{\ell+1} - w_{\ell})$$

The integrand of B_9 contains h in the forms

$$h^{(2)}(\tau,\xi,\eta) - h^{(2)}(t,x,y) \text{ and } h^{(3)}(\tau,\xi,\eta) - h^{(3)}(t,x,y)$$

The contributions with arguments (t,x,y) are evaluated separately but for the same pivotal points as $h^{(2)}(\tau,\xi,\eta)$. One has

$$h^{(2)}(t,x_{ij},y_{ij}) = [0,1,0,0,0,0]A\begin{bmatrix} h_1(t) \\ \vdots \\ h_7(t) \end{bmatrix}$$

$$h^{(3)}(t,x_{ij},y_{ij}) = [0,0,1,0,0,0]A\begin{bmatrix} h_1(t) \\ \vdots \\ h_7(t) \end{bmatrix}$$

For these expressions the retardation is zero. These contributions are accumulated in $C_{ij,\bar{m}(\ell),\bar{n}(\ell),0}$, $\ell = 1...7$.

The expression $\bar{B}_8$ (Eq. (6)) is a contour integral around the area formed by the six triangles adjacent to the control point (ij). It is evaluated separately for the sides (ℓ, ℓ+1) of the individual triangles. Because point 1 is given one has

$$\xi = x_{ij} \quad , \quad \eta = y_{ij}$$

$$\eta - y = \bar{\eta} \quad , \quad \xi - x = \bar{\xi}$$

Again the transformation (Eq. (33)) is carried out. The side (ℓ, ℓ+1) is a line u = const. Therefore,

$$d\xi = d\bar{\xi} = dv \sin\alpha \quad , \quad d\eta = \bar{\eta} = dv \cos\alpha$$

$$\rho = (u_\ell{}^2 + v^2)^{1/2}$$

The variable of integration in $\bar{B}_8$ is now dv, the limits are v_ℓ and $v_{\ell+1}$. By setting

$$v_\ell = (v_{\ell+1} + v_\ell)/2 + \bar{v}(v_{\ell+1} - v_\ell)/2$$

one obtains -1 and +1 for the limits in $\bar{v}$, and

$$\frac{dv}{d\bar{v}} = (v_{\ell+1} - v_\ell)/2$$

For an area adjacent to a control point next to the leading edge, one proceeds as follows. At the leading edge the potential is zero; there no control points are needed. The solution for steady flow suggests that the potential has a behavior as $u^{1/2}$ where u is a coordinate normal to the leading edge. (This idea had already been used for distant triangles). For control points with i = 1 (next to the leading edge) we then proceed as follows. One determines

$$\xi_\ell = x_{\bar{m}(\ell),\bar{n}(\ell)} \quad , \quad \eta_\ell = y_{\bar{m}(\ell),\bar{n}(\ell)} \qquad \ell = 1\ldots7$$

with $\bar{m}_\ell$ and $\bar{n}_\ell$ by Eq. (10) (Figure 4). Now the origin is shifted to point 7.

$$\begin{aligned} \bar{\xi}_\ell &= \xi_\ell - \xi_7 \\ \bar{\eta}_\ell &= \eta_\ell - \eta_7 \end{aligned} \qquad \ell = 1\ldots6$$

Next, a Cartesian uv-system is introduced with the u-axis pointing downstream and normal to the leading edge.

$$\cos\alpha = \bar{\eta}_6/(\bar{\xi}_6^{\,2} + \bar{\eta}_6^{\,2})^{1/2} \quad , \quad \sin\alpha = \bar{\xi}_6/(\bar{\xi}_6^{\,2} + \bar{\eta}_6^{\,2})^{1/2}$$

$$\bar{\xi} = u\cos\alpha + v\sin\alpha \quad , \quad u = \bar{\xi}\cos\alpha - \bar{\eta}\sin\alpha$$

$$\bar{\eta} = -u\sin\alpha + v\cos\alpha \quad , \quad v = \bar{\xi}\sin\alpha + \bar{\eta}\cos\alpha$$

One determines u and v for the seven points at which one wants to match h_j at those two points which lie at the leading edge h = 0. h is now represented in the form

$$h(\xi,\eta) = u^{1/2}[1,u,v,uv,v^2]\begin{bmatrix} a_1 \\ \vdots \\ a_6 \end{bmatrix}$$

In the row vector an element u^2 has been omitted for the following reason. For a configuration in which the lines 2,5 and 3,4 are parallel to 6,7, one can find an expression

$$b_1 + b_2 u + b_3 u^2$$

which vanishes for $u = u_1 = u_2 = u_5$ and $u = u_3 = u_4$ with coefficients b_1, b_2, and b_3 different from zero. If one would include the term u^2, one would obtain a degenerate matrix. If the configuration is close to the one just described, the matrix would be close to degeneracy with the consequence that some terms in the inverse might be very large.

In this case perfect matching is possible. The matching conditions are expressed as follows. One postulates

$$M \begin{bmatrix} a_1 \\ \vdots \\ a_5 \end{bmatrix} = \begin{bmatrix} h_1 \; u_1^{-1/2} \\ \vdots \\ h_5 \; u_5^{-1/2} \end{bmatrix}$$

with

$$M = \begin{bmatrix} 1 & u_1 & v_1 & u_1 v_1 & v_1^2 \\ 1 & u_2 & v_2 & u_2 v_2 & v_2^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & u_5 & v_5 & u_5 v_5 & v_5^2 \end{bmatrix}$$

One thus finds the representation

$$h(\xi,\eta) = u^{1/2}[1, u, v, uv, v^2] A \begin{bmatrix} h_1 \\ \vdots \\ h_5 \end{bmatrix}$$

where $A = M^{-1}$

Then

$h^{(2)} =$

$$\{[(1/2)u^{-1/2},(3/2)u^{1/2},(1/2)u^{-1/2}v,(3/2u^{1/2}v,(1/2)u^{-1/2}v^2)\cos\,\alpha$$

$$+\ [0,0,u^{1/2},u^{3/2},2u^{1/2}v]\sin\,\alpha\}\,A\begin{bmatrix} h_1 \\ \vdots \\ h_5 \end{bmatrix}$$

$h^{(3)} =$

$$\{-[(1/2)u^{-1/2},(3/2)u^{1/2},(1/2)u^{-1/2}v,(3/2)u^{1/2}v,(1/2)u^{-3/2}v^2]\sin\,\alpha$$

$$+\ [0,0,u^{1/2},u^{3/2},2u^{1/2}v]\cos\,\alpha\}\,A\begin{bmatrix} h_1 \\ \vdots \\ h_5 \end{bmatrix}$$

For the subsequent integrations the origin of the coordinate system is shifted to the control point. The new coordinates are denoted by $\bar{u}$ and $\bar{v}$.

$$u = u_1 + \bar{u} \qquad \bar{u} = u - u_1$$

$$v = v_1 + \bar{v} \qquad \bar{v} = v - v_1$$

For each of the six triangles a separate coordinate system $(\tilde{u},\tilde{v})$ is introduced so that the side ℓ, $\ell+1$ is a line $\tilde{u}$ = const. For $\ell = 7$ the point $(\ell+1)$ is identified with point 2.

We define for a triangle 1, ℓ, $\ell+1$; $\ell = 2 \ldots 7$

$$\cos\,\tilde{\alpha} = \frac{\bar{v}_{\ell+1} - \bar{v}_\ell}{[(\bar{u}_{\ell+1} - \bar{u}_\ell)^2 + (\bar{v}_{\ell+1} - \bar{v}_\ell)^2]^{1/2}}$$

$$\sin\,\tilde{\alpha} = \frac{\bar{u}_{\ell+1} - \bar{u}_\ell}{[(\bar{u}_{\ell+1} - \bar{u}_\ell)^2 + (\bar{v}_{\ell+1} - \bar{v}_\ell)^2]^{?}}$$

$$\bar{u} = \tilde{u} \cos \tilde{\alpha} + \tilde{v} \sin \tilde{\alpha} \qquad \tilde{u} = \bar{u} \cos \tilde{\alpha} - \bar{v} \sin \tilde{\alpha}$$

$$\bar{v} = -\tilde{u} \sin \tilde{\alpha} + \tilde{v} \cos \tilde{\alpha} \qquad \tilde{v} = \bar{u} \sin \tilde{\alpha} + \bar{v} \cos \tilde{\alpha} \tag{34}$$

$$\tilde{w} = \tilde{v}/\tilde{u} \tag{35}$$

In the triangles 1,2,3; 1,3,4; and 1,4,5 (Figure 4) the integrand is smooth and the same procedure as for triangles at inner control points is applicable. the region of integration in the $\tilde{u}\tilde{w}$-system is already a rectangle. To obtain -1 and +1 as limits in both directions one sets

$$u = (\tilde{u}_\ell/2) + \hat{u}(\tilde{u}_\ell/2)$$

$$w = (w_{\ell+1} + w_\ell)2 + \hat{w}(w_{\ell+1} - w_\ell)/2$$

Then one has

$$\frac{\partial(\xi,\eta)}{\partial(\hat{u},\hat{w})} = \tilde{u}(1/4)\tilde{u}_\ell(w_{\ell+1} - w_\ell)$$

For the triangle 1,6,7 the transformation (Eq. (34)) gives

$$\tilde{u} = -\bar{u}$$

$$\tilde{v} = -v$$

As before,

$$w = \tilde{v}/\tilde{u}$$

The limits for $\tilde{u}$ are 0 and u_1, and the limits for w are w_6 and w_7. In the representation for $h^{(2)}$ and $h^{(3)}$ the factor $u^{-1/2} = (u_1 - \tilde{u})^{-1/2}$ occurs. It is counteracted by the transformation

$$p^2 = (u_1 - \tilde{u})$$

$$2p\,dp = -d\tilde{u}$$

Then

$$\frac{\partial(\xi,\eta)}{\partial(p,w)} = -2p\tilde{u}$$

The limits for p are $u_1^{1/2}$ (lower limit) and 0 (upper limit). To transform in both directions to limits −1 and +1 we set

$$p = (u_1^{1/2})/2 - \hat{p}(u_1^{1/2}/2)$$

$$w = (w_7 + w_6)/2 + \hat{w}(w_7 - w_6)/2$$

Then

$$\frac{\partial(\xi,\eta)}{\partial(\hat{p},\hat{w})} = (1/2)p\tilde{u}\ u_1^{1/2}(w_7 - w_6)$$

For the triangle 1,5,6 one again carries out the transformation to $\tilde{u}$ and $\tilde{w}$, Eqs. (34) and (35), one has

$$\frac{\partial(\xi,\eta)}{\partial(\tilde{u},\tilde{w})} = \tilde{u}$$

The limits of integration are

$$\tilde{w}_{lower} = \tilde{w}_5 \quad , \quad \tilde{w}_{upper} = \tilde{w}_6$$

$$\tilde{u}_{lower} = 0 \quad , \quad \tilde{u}_{upper} = \tilde{u}_5 = \tilde{u}_6$$

Notice that $\sin\tilde{\alpha} < 0$ because $\tilde{u}_6 < \tilde{u}_5$. At point 6, $u = 0$, but

$$u = u_1 + \tilde{u}(\cos\tilde{\alpha} + \tilde{w}\sin\tilde{\alpha})$$

The expressions for $h^{(2)}$ and $h^{(3)}$ contain a factor $u^{-1/2}$, which introduces a singularity at point 6. To obtain a smooth integrand we replace the variable of integration $\tilde{u}$ by a new variable p

$$p^2 = u_1 + \tilde{u}(\cos\tilde{\alpha} + \tilde{w}\sin\tilde{\alpha})$$

$$2p\,dp = d\tilde{u}(\cos\tilde{\alpha} + \tilde{w}\sin\tilde{\alpha})$$

Then

$$\frac{\partial(\xi,\eta)}{\partial(p,w)} = \frac{2p\tilde{u}}{\cos\tilde{\alpha} + \tilde{w}\sin\tilde{\alpha}}$$

The transformation fails if the denominator in the Jacobian vanishes, that is, for

$$\tilde{w} = -\cot\tilde{\alpha}$$

Then, according to Eq. (35),

$$\tilde{v}\sin\tilde{\alpha} + \tilde{u}\cos\tilde{\alpha} = 0$$

and from Eqs. (34)

$$\bar{u} = 0$$

This value of $\tilde{u}$ lies within the region of integration. Let

$$\tilde{w}_{10} = (\tilde{w}_6 - \cot\tilde{\alpha})/2$$

We divide the region of integration over $\tilde{w}$ into two regions: $\tilde{w}_5 < \tilde{w} < \tilde{w}_{10}$ and $\tilde{w}_{10} < \tilde{w} < \tilde{w}_6$. The integration is carried out in the first region in terms of $\tilde{u}$ and $\tilde{w}$; in the second one in terms of p and $\tilde{w}$.

In the region $\tilde{w}_{10} < \tilde{w} < \tilde{w}_6$, the limit of p corresponding to the lower limit of $\tilde{u}$, namely $\tilde{u} = 0$, is $p = u_1^{1/2}$. The limit of p corresponding to the upper limit $\tilde{u} - \tilde{u}_6$ is

$$p_{upper} = [u_1 + \tilde{u}_6(\cos\tilde{\alpha} + \tilde{w}\sin\tilde{\alpha})]^{1/2} \tag{36}$$

It depends upon $\tilde{w}$.

One has

$$u_6 = 0 = u_1 + \tilde{u}_6(\cos \alpha + \tilde{w}_6 \sin \alpha)$$

Therefore u_1 can be replaced by

$$u_1 = \tilde{u}_6(\cos \alpha + \tilde{w}_6 \sin \alpha)$$

Substituting this form of u_1 into the expression (36) for the upper limit, one obtains

$$p_{upper} = \tilde{u}_6^{1/2}(\tilde{w}_6 - \tilde{w})^{1/2}(-\sin \tilde{\alpha})^{1/2}$$

(We had observed above that $\sin \tilde{\alpha} < 0$.)

We have generated a smooth integrand, but the boundary of the region has a singularity for $\tilde{w} = \tilde{w}_6$. To counteract the singularity in the limit of $\tilde{w}$ we introduce

$$q^2 = \tilde{w}_6 - \tilde{w} \quad , \quad \tilde{w} = \tilde{w}_6 - q^2$$

$$2q\,dq = -d\tilde{w}$$

Then

$$\frac{\partial(\xi,\eta)}{\partial(p,q)} = \frac{4pq\tilde{u}}{\cos \tilde{\alpha} + \tilde{w} \sin \tilde{\alpha}}$$

The limits for p are $p_{lower} = u_1^{1/2}$, $p_{upper} = \tilde{u}_6^{1/2} q(-\sin \tilde{\alpha})^{1/2}$. The limits for q are $q_{lower} = (\tilde{w}_6 - \tilde{w}_{10})^{1/2}$, $q_{upper} = 0$. To obtain limits ±1 in both directions we set

$$p = (1/2)(u_1^{1/2} + u_6^{1/2} q(-\sin \tilde{\alpha})^{1/2}) + \bar{p}(\tilde{u}_6^{1/2} q(-\sin \tilde{\alpha})^{1/2} - u_1^{1/2})$$

$$q = (1/2)(\tilde{w}_6 - \tilde{w}_{10})^{1/2} - (1/2)\bar{q}(\tilde{w}_6 - \tilde{w}_{10})^{1/2}$$

Then

$$\frac{\partial(\xi,\eta)}{\partial(\bar{p},\bar{q})} = \frac{pq\tilde{u}}{\cos \alpha + w \sin \alpha}\,[\tilde{u}_6^{1/2} q(-\sin \tilde{\alpha})^{1/2} - u_1^{1/2}][\tilde{w}_6 - \tilde{w}_{10}]^{1/2}$$

The integration is now carried out in terms of $\bar{p}$ and $\bar{q}$.

For $\tilde{w}_5 < \tilde{w} < \tilde{w}_{10}$, one has a rectangle on the $\tilde{u},\tilde{w}$-plane. To obtain limits -1 and +1, we set

$$\tilde{u} = \frac{\tilde{u}_6}{2} + \hat{u}\,\frac{\tilde{u}_6}{2}$$

$$\tilde{w} = \frac{\tilde{w}_{10} + \tilde{w}_5}{2} + \hat{w}\,\frac{\tilde{w}_{10} - \tilde{w}_5}{2}$$

Then

$$\frac{\partial(\xi,\eta)}{\partial(\hat{u},\hat{w})} = \frac{\tilde{u}}{4}\,\tilde{u}_6(\tilde{w}_{10} - \tilde{w}_5)$$

The integration is carried out in terms of $\hat{u}$ and $\hat{w}$.

An analogous procedure is carried out in the triangle 1,7,2. The critical term is again $u^{1/2}$ where,

$$u = u_1 + \tilde{u}(\cos\tilde{\alpha} + \tilde{w}\sin\tilde{\alpha})$$

the factor of $\tilde{u}$ vanishes for $\tilde{w} = -\cot\tilde{\alpha}$. (Here $\sin\tilde{\alpha} > 0$.) The region of integration in the $\tilde{w}$ direction is divided. Let

$$\tilde{w}_{10} = (\tilde{w}_7 - \cot\tilde{\alpha})/2$$

If $\tilde{w}_7 > \tilde{w}_2$ (which is unlikely), then there is only one region of integration $\tilde{w}_7 < \tilde{w} < \tilde{w}_{10}$ and $\tilde{w}_{10} < \tilde{w} < \tilde{w}_2$. In the first region we introduce, as before, instead of u

$$p^2 = u_1 + \tilde{u}(\cos\tilde{\alpha} + \tilde{w}\sin\tilde{\alpha})$$

The limits for p are

$$p_{lower} = u_1^{1/2}$$

$$p_{upper} = [u_1 + \tilde{u}_7(\cos\tilde{\alpha} + \tilde{w}\sin\tilde{\alpha}]^{1/2}$$
$$= \tilde{u}_7^{1/2}(\sin\tilde{\alpha})^{1/2}(\tilde{w} - \tilde{w}_7)^{1/2}$$

Next, one removes the singularity in the upper limit:

$$q^2 = \tilde{w} - \tilde{w}_7$$

Then $q_{lower} = 0$ and $q_{upper} = (\tilde{w}_{10} - \tilde{w}_7)^{1/2}$. A further transformation

$$q = (1/2)(\tilde{w}_{10} - \tilde{w}_7)^{1/2} + \hat{q}(1/2)(\tilde{w}_{10} - \tilde{w}_7)^{1/2}$$

$$p = (1/2)[u_1^{1/2} + \tilde{u}_7^{1/2}(\sin\tilde{\alpha})]^{1/2}q$$
$$+ \hat{p}(1/2)[\tilde{u}_7^{1/2}(\sin\tilde{\alpha})^{1/2}q - u^{1/2}]$$

gives in both directions of the $\hat{p},\hat{q}$-plane the limits are -1 and +1.

$$\frac{\partial(\xi,\eta)}{\partial(\hat{p},\hat{q})} = \frac{pq\tilde{u}}{\cos\tilde{\alpha} + \tilde{w}\sin\tilde{\alpha}}[\tilde{u}_7^{1/2}(\sin\tilde{\alpha})^{1/2}q - u_1^{1/2}][\tilde{w}_{10} - w_7]^{1/2}$$

In the region $\tilde{w}_{10} < \tilde{w} < \tilde{w}_2$ we introduce $\hat{u}$ and $\hat{w}$ by

$$\tilde{u} = (1/2)\tilde{u}_7 + \hat{u}(1/2)\tilde{u}_7$$
$$\tilde{w} = (1/2)(\tilde{w}_2 + \tilde{w}_{10}) + \hat{w}(1/2)(\tilde{w}_2 - \tilde{w}_{10})$$

In the $\hat{u},\hat{w}$-plane the limits are -1 and +1 in both directions. One has

$$\frac{\partial(\xi,\eta)}{\partial(\hat{u},\hat{q})} = (\tilde{u}/4)\tilde{u}_7(\tilde{w}_2 - \tilde{w}_{10})$$

For control points immediately upstream of the trailing edge ($i=i_t-1$), one determines, as for other hexagons

$$\xi_\ell = x_{\bar{m}(\ell),\bar{n}(\ell)} \qquad \ell=1\ldots7$$

$$\eta_\ell = (y_{\bar{m}(\ell),\bar{n}(\ell)})$$

Next, one shifts the coordinate system to point 3

$$\bar{\xi}_\ell = \xi_\ell - \xi_\ell$$

$$\bar{\eta}_\ell = \eta_\ell - \eta_3$$

The sweep angle at the trailing edge is given by

$$\cos\alpha = \bar{\eta}_4/(\bar{\xi}_4{}^2 + \bar{\eta}_4{}^2)^{1/2}, \qquad \sin\alpha = \bar{\xi}_4/(\bar{\xi}_4{}^2 + \bar{\eta}_4{}^2)^{1/2}$$

A coordinate system is introduced so that the trailing edge is the line $u = 0$

$$\bar{\xi} = u\cos\alpha + v\sin\alpha \quad , \qquad u = \bar{\xi}\cos\alpha - \bar{\eta}\sin\alpha$$

$$\bar{\eta} = -u\sin\alpha + v\cos\alpha \quad , \qquad v = \bar{\xi}\sin\alpha + \bar{\eta}\cos\alpha$$

Next one determines the values of u_ℓ, v_ℓ, $\ell = 1\ldots7$, $(u_3 = 0, u_4 = 0)$. The steady problem suggests a representation for h of the form

$$h(\xi,\eta,\eta) = [1, u, v, uv, v^2, (-u)^{3/2}] \begin{bmatrix} a_1(\tau) \\ \vdots \\ a_7(\bar{\tau}) \end{bmatrix}$$

For the reason given in conjunction with the presentation of h in the vicinity of leading edge, the term u^2 is not included in the row vector on the left.

In the present form, only 6 functions of u and v and 6 coefficients a_ℓ are available; therefore, only an approximate matching is possible. The procedure is analogous to that for an inner control point. One introduces a 7 by 6 matrix

$$M = \begin{bmatrix} 1, & u_1, & v_1, & u_1v_1, & v_1^2, & (-u_1)^{3/2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1, & u_\ell, & v_\ell, & u_\ell v_\ell, & v_\ell^2, & (-u_\ell)^{3/2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1, & u_7, & v_7, & u_7, v_7, & v_7^2, & (-u_7)^{3/2} \end{bmatrix}$$

and forms

$$A = [M^+M]^{-1}M^+$$

One thus obtains the approximation

$$h(\tau,\xi,\eta) = [1,u,v,uv,\ v,^2(-u)^{3/2}]\ A \begin{bmatrix} h_1(\tau) \\ \vdots \\ h_7(\tau) \end{bmatrix}$$

Hence,

$$h^{(2)}(\tau,\xi,\eta) = \{[0,1,0,v,0,^{-3/2}(-u)^{1/2}]\cos\alpha$$

$$+ [0,0,1,u,2v,0]\sin\alpha]\}\ A \begin{bmatrix} h_1(\tau) \\ \vdots \\ h_7(\tau) \end{bmatrix}$$

and

$$h^{(3)}(\tau,\xi,\eta) = \{-[0,1,0,v,0\ -(3/2)(-u)^{1/2}]\sin\alpha$$

$$+ [0,0,1,u,2v,0]\cos\alpha]\}\quad A \begin{bmatrix} h_1(\tau) \\ \vdots \\ h_7(\tau) \end{bmatrix}$$

For the integration the origin of the coordinate is transferred to the point 1, (notice that $u_1<0$)

$$\bar{u} = u - u_1 \qquad u = u_1 + \bar{u}$$

$$\bar{v} = v - v_1 \qquad v = v_1 + \bar{v}$$

The six triangles 1, ℓ, ℓ+1 are treated separately. In each triangle a coordinate system is introduced so that $u_\ell = u_{\ell+1}$:

$$\cos\tilde{\alpha} = (\bar{u}_{\ell+1} - \bar{u}_\ell) / [(\bar{u}_{\ell+1} - \bar{u}_\ell)^2 + (\bar{v}_{\ell+1} - \bar{v}_\ell)^2]^{1/2}$$

$$\sin\tilde{\alpha} = (\bar{v}_{\ell+1} - \bar{v}_\ell) / [(u_{\ell+1} - \bar{u}_\ell)^2 + (\bar{v}_{\ell+1} - \bar{v}_\ell)^2]^{1/2}$$

$$\bar{u} = \tilde{u}\cos\tilde{\alpha} + \tilde{v}\sin\tilde{\alpha} \quad , \qquad \tilde{u} = \bar{u}\cos\tilde{\alpha} - \bar{v}\sin\tilde{\alpha}$$

$$\bar{v} = -\tilde{u}\sin\tilde{\alpha} + \tilde{v}\cos\tilde{\alpha} \quad , \qquad \tilde{v} = \bar{u}\sin\tilde{\alpha} + \bar{v}\cos\tilde{\alpha}$$

One determines $\tilde{u}_\ell = \tilde{u}_{\ell+1}$, $\tilde{v}_\ell$, and $\tilde{v}_{\ell+1}$.

Next one introduces

$$\tilde{w} = \tilde{v}/\tilde{u}$$

and determines $\tilde{w}_\ell$ and $\tilde{w}_{\ell+1}$. Then

$$\frac{\partial(\xi,\eta)}{\partial(\tilde{u},\tilde{w})} = \tilde{u}$$

In the triangles 156, 167, and 172 the integrand is regular. The region of integration for these triangles in the $\tilde{u}\tilde{w}$ plane is a rectangle. To obtain limits -1 and +1, one introduces new coordinates $\hat{u}$ and $\hat{w}$:

$$\tilde{u} = \frac{\tilde{u}_\ell}{2} + \hat{u}\,\frac{\tilde{u}_\ell}{2}$$

$$\tilde{w} = \frac{(\tilde{w}_\ell + \tilde{w}_{\ell+1})}{2} + \hat{w}\,\frac{\tilde{w}_{\ell+1} + \tilde{w}}{2}$$

Then

$$\frac{\partial(\xi,\eta)}{\partial(\hat{u},\hat{w})} = \frac{\tilde{u}}{4}\quad \tilde{u}_\ell (w_{\ell+1} - w_\ell)$$

For the triangle 134, $\tilde{u} = \bar{u}$, $\tilde{v} = \bar{v}$. In $h^{(2)}$ and $h^{(3)}$, a term $(-u)^{1/2}$ occurs, which introduces a singularity into the integrand. One has

$$-u = -(u_1 + \bar{u}).$$

Therefore, we set

$$p^2 = -(u_1 + \bar{u})$$

$$2pdp = -\,d\bar{u}$$

Then

$$\frac{\partial(\xi,\eta)}{\partial(p,w)} = -2\bar{u}p.$$

The limits for p are

$$p_{lower} = (-u_1)^{1/2}$$

$$p_{upper} = 0.$$

Limits ±1 are obtained by introducing

$$p = (1/2)(-u_1)^{1/2} - (\hat{p}/2)(-u_1)^{1/2}$$

$$\tilde{w} = 1/2\ (w_3 + w_4) + (\hat{w}/2)(w_4 - w_3).$$

Then

$$\frac{\partial(\xi,\eta)}{\partial(\rho,w)} = \frac{\tilde{u}p}{2} (-u_1)^{1/2}(\tilde{w}_4 - \tilde{w}_3)$$

For the triangle 1, 2, 3, one has after the initial transformation

$$(-u) = -\bar{u}_1 - \bar{u} = -u_1 - \tilde{u}\,(\cos\tilde{\alpha} + w\sin\tilde{\alpha})$$

here $\sin\tilde{\alpha} > 0$, since $\bar{u}_3 > \bar{u}_2$.

To counteract the singularity introduced by $(-u)^{1/2}$, we set

$$p^2 = (-u) = -u_1 - \tilde{u}\,(\cos\tilde{\alpha} + \tilde{w}\sin\tilde{\alpha})$$

Here $\sin\tilde{\alpha} > 0$ because $\tilde{u}_3 > \tilde{u}_2$. The transformation fails for

$$\tilde{w} = -\cot\tilde{\alpha}.$$

Let

$$\tilde{w}_{10} = (\tilde{w}_3 - \cot\tilde{\alpha})/2.$$

If $\tilde{w}_2 > \tilde{w}_{10}$, there is only one region of integration

$$\tilde{w}_2 < \tilde{w} < \tilde{w}_3.$$

If $\tilde{w}_2 < \tilde{w}_{10}$, one has the two regions

$$\tilde{w}_2 < \tilde{w} < w_{10} \text{ and } \tilde{w}_{10} < \tilde{w} < \tilde{w}_3.$$

For $\tilde{w}_2 < \tilde{w} < \tilde{w}_{10}$, the integrand is smooth and the integration is carried out in terms of $\tilde{u}$ and $\tilde{w}$. To obtain limits ± 1, we introduce instead of $\tilde{u}$ and $\tilde{w}$,

$$\tilde{u} = \tilde{u}(\tilde{u}_2/2) + \hat{u}(\tilde{u}_2/2)$$

$$\tilde{w} = (\tilde{w}_{10} + \tilde{w}_2)/2 + \hat{w}(w_{10} - \tilde{w}_2)/2)$$

Then, for $\tilde{w}_2 < \tilde{w} < \tilde{w}_{10}$

$$\frac{\partial(\xi_*,\eta_*)}{\partial(u,w)} = \frac{\tilde{u}}{4}\,\tilde{u}_2(\tilde{w}_{10}-\tilde{w}_2).$$

For $\tilde{w}_{10} < \tilde{w} < \tilde{w}_3$, p is introduced by the above formula. As limits of integration for p, one finds

$$p_{lower} = (-u_1)^{1/2}$$

$$p_{upper} = (-u_1 - \tilde{u}_3(\cos\tilde{\alpha} + \tilde{w}\sin\tilde{\alpha}))^{1/2}$$

But since u = 0 for $\tilde{u} = \tilde{u}_3$, $\tilde{w} = \tilde{w}_3$, one has $-u_1 = \tilde{u}_3$, and then from the transformation formulae

$$-u_1 = \tilde{u}_3(\cos\tilde{\alpha} + \tilde{w}_3\sin\tilde{\alpha})$$

Therefore,

$$p_{upper} = (\tilde{u}_3^{1/2}\ (\sin\tilde{\alpha}))^{1/2}(\tilde{w}_3 - \tilde{w})^{1/2}.$$

Let

$$\tilde{w}_3 - \tilde{w} = q^2$$

$$q_{lower} = (\tilde{w}_3 - \tilde{w}_{10})^{1/2} \qquad \text{for } \tilde{w}_{10} > \tilde{w}_2$$

$$= (\tilde{w}_3 - \tilde{w}_2)^{1/2} \qquad \text{for } \tilde{w}_{10} < \tilde{w}_2$$

$$q_{upper} = 0$$

To obtain limits ±1, we introduce new variables p and q by

$$p = (1/2)[\tilde{u}_3^{1/2}(\sin\tilde{\alpha})^{1/2}q+(-u_1)^{1/2}]+(1/2)\hat{p}[\hat{u}_3^{1/2}(\sin\tilde{\alpha})^{1/2}q-(-u_1)^{1/2}]$$

$$q = (1/2)(\tilde{w}_3 - \tilde{w}_{10})^{1/2} - (1/2)\hat{q}\,(\tilde{w}_3 - \tilde{w}_{10})^{1/2}$$

Then,

$$\frac{\partial(\xi,\eta)}{\partial(p,q)} = \frac{\tilde{u}\,\hat{p}\,\hat{q}}{\cos\tilde{\alpha}+\tilde{w}\sin\tilde{\alpha}} = [(-\tilde{u})^{1/2} - \tilde{u}_3^{1/2}(\sin\tilde{\alpha})^{1/2}q][\tilde{w}_3-\tilde{w}_{10})^{1/2}]$$

If $\hat{w}_2 > \hat{w}_{10}$, one replaces in these formulae $\tilde{w}_{10}$ by $\tilde{w}_2$.

For the triangle 1, 4, 5, one introduces

$$\tilde{w}_{10} = (\tilde{w}_4 - \cot\tilde{\alpha})/2$$

and obtains regions of integration

$$\tilde{w}_4 < \tilde{w} < \tilde{w}_5 \qquad \text{for } \hat{w}_{10} > \hat{w}_5$$

and

$$\tilde{w}_4 < \tilde{w} < \tilde{w}_{10} \text{ and } \tilde{w}_{10} < \tilde{w} < \tilde{w}_5 \qquad \text{for } \tilde{w}_{10} < \tilde{w}_5$$

For the region $\tilde{w}_{10} < \tilde{w} < \tilde{w}_5$ (which occurs in the second case), one obtains limits ±1 by introducing new coordinates $\hat{u}$, $\hat{w}$

$$\tilde{u} = (\tilde{u}_5/2) + \hat{u}(\tilde{u}_5/2)$$

$$\tilde{w}_2 = (\tilde{w}_5 + \tilde{w}_{10})/2 + \hat{w}(\tilde{w}_5 - \tilde{w}_{10})/2$$

One has

$$\frac{\partial(\xi,\eta)}{\partial(\hat{u},\hat{w})} = \frac{\tilde{u}}{4}\,\tilde{u}_5(\tilde{w}_5 - \tilde{w}_{10})$$

In the regions

$$\tilde{w}_4 < \tilde{w} < \tilde{w}_{10}$$

One sets

$$p^2 = (-u_1) - \tilde{u}\,(\cos\tilde{\alpha} + \tilde{w}\sin\tilde{\alpha}).$$

Here $\sin\tilde{\alpha} < 0$. The limits of p are

$$p_{lower} = (-u_1)^{1/2}$$

$$p_{upper} = [(-u_1 - \tilde{u}_4(\cos\tilde{\alpha} + \tilde{w}\sin\tilde{\alpha})]^{1/2} = \tilde{u}_4^{1/2}(-\sin\tilde{\alpha})^{1/2}(\tilde{w} - \tilde{w}_4)^{1/2}$$

The form of p_{upper} suggests that one introduce

$$q^2 = (\tilde{w} - \tilde{w}_4)$$

then

$$q_{lower} = 0$$

$$q_{upper} \text{ is } (\tilde{w}_{10} - \tilde{w}_4)^{1/2} \quad \text{for } \tilde{w}_{10} < \tilde{w}_5$$

$$(\tilde{w}_{10} - \tilde{w}_5)^{1/2} \quad \text{for } \tilde{w}_{10} > \tilde{w}_5$$

One introduces new variables $\hat{p}$ and $\hat{q}$ to obtain limits ±1

$$p=(1/2)[\tilde{u}_4^{1/2}(-\sin\tilde{\alpha})^{1/2}q+(-u_1)^{1/2}]+(1/2)\hat{p}[\tilde{u}_4^{1/2}(-\sin\tilde{\alpha})^{1/2}q-(-u_1)^{1/2}]$$

$$q=(1/2)([\tilde{w}_{10} - \tilde{w}_4)^{1/2} + (1/2)\hat{q}\,[\tilde{w}_{10} - \tilde{w}_4)^{1/2}]$$

then

$$\frac{\partial(\xi,\eta)}{\partial(\hat{p},\hat{q})} = \frac{\tilde{u}\,\hat{\rho}\,\hat{q}}{\cos\tilde{\alpha}+\tilde{w}\sin\tilde{\alpha}}\;[(-\tilde{u}_1)^{1/2} - \tilde{u}_4^{1/2}(-\sin\tilde{\alpha})^{1/2}q][\tilde{w}_{1\bar{0}}\tilde{w}_4)]^{1/2}$$

If $\tilde{w}_{10} < \tilde{w}_5$, one applies the above formulae with $\tilde{w}_{10}$ replaced by $\tilde{w}_5$.

For a control point at the trailing edge $(i = i_t)$, (Figure 12), one first determines

$$\xi_\ell = x_{\bar{m}(\ell),\,\bar{n}(\ell)}$$

$$\eta_\ell = y_{\bar{m}(\ell),\,\bar{n}(\ell)}$$

In introducing a triangulation of the wake, it facilitates the determination of h if one lets $\eta_3=\eta_2$, $\eta_4 = \eta_1$. This means that the basic quadrangles are trapezoids. Next we shift the coordinate systems into point 1

$$\bar{\xi}_\ell = \xi_\ell - \xi_1$$
$$\ell = 2....7$$
$$\bar{\eta}_\ell = \eta_\ell - \eta_1$$

One introduces a Cartesian system u, v, where u is perpendicular to the trailing edge

$$\cos\alpha = (\bar{\eta}_5-\bar{\eta}_2)/[(\bar{\xi}_5-\bar{\xi}_2)^2+(\bar{\eta}_5-\bar{\eta}_2)^2]^{1/2}$$

$$\sin\alpha = (\bar{\xi}_5-\bar{\xi}_2)/[(\bar{\xi}_5-\bar{\xi}_2)^2+(\bar{\eta}_5-\bar{\eta}_2)^2]^{1/2}$$

$$\bar{\xi} = u\cos\alpha + v\sin\alpha \quad , \quad u = \bar{\xi}\cos\alpha - \bar{\eta}\sin\alpha$$

$$\bar{\eta} = -u\sin\alpha + v\cos\alpha \quad , \quad v = \bar{\xi}\sin\alpha + \bar{\eta}\cos\alpha$$

The steady problem suggests a representation

$$h = [1,\ u,\ v,\ uv,\ v^2,\ 0] \begin{bmatrix} a_1(\tau) \\ \vdots \\ a_6(\tau) \end{bmatrix}$$

for the wake, and $h = [1,\ u,\ v,\ uv,\ v^2(-u)^{3/2}]$ for the wing.

A term u^2 has not been included for reasons given above. Again, only approximate matching is possible. The procedure is analogous to that for an inner control point. One substitutes u and v for points 1 to 7 in the expressions applicable either for the wings or the wake, and obtains a 7 by six matrix M

$$
M = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & v_2 & 0 & v_2^2 & 0 \\
1 & u_3 & v_3 & u_3 v_3 & v_3^2 & 0 \\
1 & u_4 & v_4 & u_4 v_4 & v_4^2 & 0 \\
1 & u_5 & v_5 & u_5 v_5 & v_5^2 & 0 \\
1 & u_6 & v_6 & u_6 v_6 & v_6^2 & (-u_6)^{3/2} \\
1 & u_7 & v_7 & u_7 v_7 & v_7^2 & (-u_7)^{3/2}
\end{bmatrix}
$$

Then one forms

$$A = [M+M]^{-1} M+$$

and obtains the approximation

$$\begin{bmatrix} a_1(\tau) \\ \vdots \\ a_7(\tau) \end{bmatrix} = A \begin{bmatrix} h_1(\tau) \\ \vdots \\ h_7(\tau) \end{bmatrix}$$

Then, in the wake

$$h^{(2)} = \{[0,1,0,v,0,0]\cos\alpha + [0,0,1,u,2v,0]\sin\alpha\} A \begin{bmatrix} h_1(\tau) \\ \vdots \\ h_7(\tau) \end{bmatrix}$$

and, at the wing

$$h_2^{(2)}=\{[0,1,0,v,0,-(3/2)(-u)^{1/2}]\cos\alpha+[0,0,1,u,2v,0]\sin\alpha]\}A\begin{bmatrix}h_1(\tau)\\ \vdots\\ h_7(\tau)\end{bmatrix}$$

Furthermore, in the wake

$$h_2^{(3)}=\{-[0,1,0,v,0,0]\sin\alpha+[0,0,1,u,2v,0]\cos\alpha]\}\ A\begin{bmatrix}h_1(\tau)\\ \vdots\\ h_7(\tau)\end{bmatrix}$$

and at the wing

$$h^{(3)}=\{-[0,1,0,v,0,-(3/2)(-u)^{1/2}]\sin\alpha+[0,0,1,u,2v,0]\cos\alpha]\}A\begin{bmatrix}h_1(\tau)\\ \vdots\\ h_7(\tau)\end{bmatrix}$$

Next one considers individual triangles and rotates the uv system into $\tilde{u}\tilde{v}$ systems so that the sides ℓ, $\ell+1$ are lines $\tilde{u}$ = const.

$$\cos\tilde{\alpha} = (v_{\ell+1} - v_\ell)|[u_{\ell+1} - u_\ell)^2 + (v_{\ell+1} - v_\ell)^2]^{1/2}$$

$$\sin\tilde{\alpha} = (u_{\ell+1} - u_\ell)|[u_{\ell+1} - u_\ell)^2 + (v_{\ell+1} - v_\ell)^2]^{1/2}$$

$$u = \tilde{u}\cos\tilde{\alpha} + \tilde{v}\sin\alpha \quad , \quad \tilde{u} = u\cos\alpha - v\sin\alpha$$

$$v = -\tilde{u}\sin\tilde{\alpha} + \tilde{v}\cos\alpha \quad , \quad \tilde{v} = u\sin\alpha + v\cos\alpha$$

Moreover, $\tilde{w} = \dfrac{\tilde{v}}{\tilde{w}}$.

In the triangles 123, 134, and 145, the integrands are regular. To obtain limits of integration ±1, one introduces new variables $\hat{u}$ and $\hat{v}$ by

$$\tilde{u} = (\tilde{u}_\ell/2) + \hat{u}(\tilde{u}_\ell/2))$$

$$\tilde{w} = (\tilde{w}_{\ell+1} + \tilde{w}_\ell)/2 + \hat{w}\ (\tilde{w}_{\ell+1} - \tilde{w}_\ell)/2$$

One has

$$\frac{\partial(\xi,\eta)}{\partial(u,w)} = (1/4)\ \tilde{u}\ \tilde{u}_\ell(\tilde{w}_{\ell+1} - \tilde{w}_\ell)$$

In the triangles (156), (1,6,7), and (1,7,2), representations of h includes a term $(-u)^{1/2}$, which introduces a singularity long the line 2, 1, 5. One has

$$(-u) = -\tilde{u}(\cos\tilde{\alpha} + \tilde{w}\sin\alpha).$$

In the triangle (1,6,7), u is negative. Therefore, $\cos\tilde{\alpha} + \tilde{w}\sin\alpha < 0$

$$(-u)^{1/2} = \tilde{u}^{\ 1/2}[-(\cos\tilde{\alpha} + \tilde{w}\sin\alpha)]^{1/2}$$

We set $p^2 = \tilde{u}$, and leave $\tilde{w}$ unchanged. To obtain limits ±1, a $\hat{p}$, $\hat{w}$ system is introduced by

$$p = (\tilde{u}_6)^{1/2}/2 + \hat{p}(\tilde{u}_6)^{1/2}/2$$

$$\tilde{w} = (\tilde{w}_6 + \tilde{w}_5)/2 + \hat{w}\ (\tilde{w}_6 - \tilde{w}_5)/2)$$

then

$$\frac{\partial(\xi,\eta)}{\partial(p,w)} = \frac{\tilde{u}p}{2}\ (\tilde{u}_6)^{1/2}(\tilde{w}_6 - \tilde{w}_5)$$

In the triangle (1,5,6)

$$u = 0, \quad \text{for } \tilde{w} = \tilde{w}_5.$$

Therefore, $\cos\tilde{\alpha} + \tilde{w}_5 \sin\alpha = 0$, and $(-u) = -\tilde{u}((\tilde{w} - \tilde{w}_5)\sin\tilde{\alpha}$. Then one introduces

$$q^2 = (\tilde{w} - \tilde{w}_5)$$

and subsequently, $\hat{q}$ and $\hat{u}$ by

$$q = ((\tilde{w}_6 - \tilde{w}_5)^{1/2}/2) + \hat{q}(\tilde{w}_6 - \tilde{w}_5)^{1/2}/2$$

$$u = (\tilde{u}_5/2) + \hat{u}\ (\tilde{u}_5)/2.$$

Then,

$$\frac{\partial(\xi, \eta)}{\partial(\rho, q)} = \tilde{u}q\ (1/2)\tilde{u}_5\ )[\tilde{w}_6 - \tilde{w}_5)^{1/2}$$

In the triangle 1,7,2, one sets

$$q^2 = (\tilde{w}_2 - \tilde{w})$$

and introduces

$$q = ((\tilde{w}_2 - \tilde{w}_7)^{1/2}/2) - \hat{q}\ (\tilde{w}_2 - \tilde{w}_7)^{1/2}/2$$

$$\tilde{u} = \tilde{u}_7/2 + \hat{u}(\tilde{u}_7/2)$$

Then

$$\frac{\partial(\xi, \eta)}{\partial(u, q)} = \tilde{u}q(1/2)\tilde{u}_7(\tilde{w}_2 - \tilde{w}_7)^{1/2}$$

For a control point at the x axis (the axis of symmetry), one has in principle, the same procedure as for an inner control point, but it is possible to take the symmetry properties of the flow field into account. Referring to Figure 13, one has

$$\bar{m}_1 = i \qquad \bar{n}_1 = 0$$

$$\bar{m}_4 = i+1 \qquad \bar{n}_4 = 0$$

$$\bar{m}_5 = i \qquad \bar{n}_5 = 1$$

$$\bar{m}_6 = i-1 \qquad \bar{n}_6 = 1$$

$$\bar{m}_7 = i-1 \qquad \bar{n}_7 = 0$$

$$\xi_\ell = x_{\bar{m}(\ell),\bar{n}(\ell)}, \quad \eta_\ell = y_{\bar{m}(\ell),\bar{n}(\ell)} \qquad \ell = 1,\ 4,\ 5,\ 6,\ 7$$

Moreover

$$\xi_2 = \xi_6, \qquad \eta_2 = -\eta_6$$

$$\xi_3 = \xi_5 \qquad \eta_3 = -\eta_5$$

The origin of a $\bar{\xi},\bar{\eta}$ system is placed into point 1

$$\bar{\xi}_\ell = \xi_\ell - \xi_1, \qquad \bar{\eta}_\ell = \eta_\ell.$$

For a flow field which is symmetric with respect to the x axis, one sets

$$h = [1,\ \bar{\xi},\ \bar{\xi}^2,\ \bar{\eta}^2] \begin{bmatrix} a_1 \\ \vdots \\ a_4 \end{bmatrix}$$

To match at five points 1, 4, 5, 6, and 7, one introduces a matrix

$$
M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \xi_4 & \xi_4^2 & \eta_4^2 \\ 1 & \xi_5 & \xi_5^2 & \eta_5^2 \\ 1 & \xi_6 & \xi_6^2 & \eta_6^2 \\ 1 & \xi_7 & \xi_7^2 & \eta_7^2 \end{bmatrix}
$$

and forms

$$
A = [M^+M]^{-1}M^+
$$

Then one obtains the following approximation for h

$$
h = [1, \bar{\xi}, \bar{\xi}^2, \eta^2) \ A \begin{bmatrix} h_1 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \end{bmatrix}
$$

Hence,

$$
h^{(2)}_{\text{symmetric}} = [0, \ 1, \ 2\xi, 0] \ A \begin{bmatrix} h_1 \\ h_4 \\ h_5 \\ h_7 \end{bmatrix}
$$

$$
h^{(3)}_{\text{symmetric}} = [0, 0, 0, 2\eta] \ A \begin{bmatrix} h_1 \\ h_4 \\ h_5 \\ h_7 \end{bmatrix}
$$

For the antisymmetric part of h, one sets

$$h_{anti} = [u,\ \bar{\xi}\eta] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} .$$

and defines

$$M = \begin{bmatrix} \eta_5 & , & \bar{\xi}_5 \eta_5 \\ \eta_6 & , & \bar{\xi}_6 \eta_5 \end{bmatrix}$$

$$A = M^{-1}$$

now,

$$h_{anti} = [\eta,\ \bar{\xi}\eta] \quad A \begin{bmatrix} \eta_5 \\ \eta_6 \end{bmatrix}$$

$$h^{(2)}_{anti} = [0, \eta] \quad A \begin{bmatrix} \eta_5 \\ \eta_6 \end{bmatrix}$$

$$h^{(3)}_{anti} = [0, \bar{\xi}] \quad A \begin{bmatrix} \eta_5 \\ \eta_6 \end{bmatrix}$$

The steps for the individual triangles are the same as for the interior of the wing. Only the triangles in one half of the wing need to be considered. The sign of the contribution of the other triangle are determined by the symmetry cr antisymmetry.

As mentioned in Section IV, the leading edge has been modified in the vicinity of the axis of symmetry (see Figure 2). For a control point next to the leading edge, one obtains a configuration consisting of 5 triangles (Figure 14).

$$\bar{m}_1 = 1 \qquad \bar{n}_1 = 0$$

$$\bar{m}_4 = 2 \qquad \bar{n}_4 = 0$$

$$\bar{m}_5 = 1 \qquad \bar{n}_5 = 1$$

$$\bar{m}_6 = 0 \qquad \bar{n}_6 = 1$$

$$\xi_\ell = x(\bar{m}_\ell, \bar{n}_\ell), \qquad \eta_\ell = y(\bar{m}_\ell, \bar{n}_\ell), \quad \ell = 1,4,5,6$$

$$\xi_2 = \xi_6, \qquad \eta_2 = -\eta_6$$

$$\xi_3 = \xi_5 \qquad \eta_3 = -\eta_5$$

We introduce an auxiliary point 6'

$$\xi_{6'} = \xi_{6'} \quad \eta_{6'} = 0$$

and place the origin into point 6'

$$\bar{\xi}_\ell = \xi_\ell - \xi_{6'}, \quad \bar{\eta}_\ell = \eta_\ell$$

For the symmetric case, h is then represented by

$$h = (\bar{\xi}^{1/2}, \bar{\xi}^{3/2}, \bar{\xi}^{1/2}\eta^2) \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

then $h_6 = 0$.

To match at points 1, 4, and 5, one introduces

$$M = \begin{bmatrix} \bar{\xi}_1^{1/2} & \bar{\xi}_1^{3/2} & 0 \\ \bar{\xi}_4^{1/2} & \bar{\xi}_4^{3/2} & 0 \\ \bar{\xi}_5^{1/2} & \bar{\xi}_5^{3/2} & \bar{\xi}_5^{1/2} \quad \eta_\xi^2 \end{bmatrix}$$

and evaluates

$$A = M^{-1}$$

Then

$$h_{symmetric} = [\bar{\xi}^{1/2}, \bar{\xi}^{3/2}, \bar{\xi}^{1/2}\eta^2]\ A \begin{bmatrix} \\ h_4 \\ h_5 \end{bmatrix}$$

For the antisymmetric case, one has

$$h_{anti} = \bar{\xi}^{1/2}\eta(\bar{\xi}_5^{1/2}\eta_5)^{-1}$$

hence, $h^{(2)}$ and $h^{(3)}$.

After the introduction of the point 6', one deals again with a hexagon in which the point 6' plays the role of point 7. The only difference lies in the fact that the line 6'2, as well as 6,6', are part of the leading edge. The treatment of the triangles (1,4,5), (1,4,6), and (1,6,6') is the same as for a control point adjacent to the leading edge. The contributions of the remaining triangles are found immediately on the basis of symmetry consideration.

The treatment of the trailing edge is different, because there exists control points directly at the trailing edge. On the other hand, the singularity at the wing has the character of $u^{3/2}$, while the flow in the wake is free of a singularity.

The author proposes not to modify the trailing edge at the line of symmetry, and to disregard the $u^{3/2}$ singularity. For the control points at the line of symmetry, next to the trailing edge as well as at the trailing edge, h is then represented by

$$[1, \xi, \eta, \xi^2, \xi\eta, \eta^2] \quad \begin{bmatrix} a_1 \\ \vdots \\ a_7 \end{bmatrix}$$

i.e., in the same manner as any other control point in the interior or the wake. The symmetry relations can be taken into account as for other control points on the x-axis. One might experiment with other approximations, but it is likely that the

treatment of a root region will always remain somewhat unsatisfactory until one extends the method to include the fuselage and the fairings at the root sections. This was already mentioned in Section IV.

SECTION X

THE NUMBER OF PIVOTAL POINTS TO BE USED FOR THE INTEGRATION WITHIN ONE TRIANGLE

The number of pivotal points to be used for the integration depends, of course, on the variations of the integrand within this region. These variations are brought about mainly by the denominator

$$DEN = \rho[M(\xi - x) + \rho)^2$$

Here

$$\rho = [(\xi - x_{ij})^2 + (\eta - y_{ij})]^{1/2}$$

and $x = x_{ij}$, $y = y_{ij}$ are the coordinates of the control point under consideration, ξ and η are the umbral variables of integration.

It is proposed to take the ratio of the maximum to the minimum over the triangle of the expression DEN as criterion for the choice of the integration formula. The extrema occur at the contour, but not necessarily at the corners of the triangle. We, therefore, compute DEN at the corners of the triangle and at the midpoints of the sides, approximate DEN along each side by a second degree polynomial, and then determine, separately for each side, the extrema. The corners are assigned indices 1, 2, and 3 and the midpoints of the sides from ℓ to $\ell + 1$ the index $\ell + 1/2$. (For $\ell = 3$, we identify $\ell + 1$ with 1.)

The following steps are then carried out. One determines

$$\xi_\ell, \ \eta_\ell; \qquad \ell = 1, 2, 3$$

$$\xi_{\ell+1/2} = (1/2)(\xi_\ell + \xi_{\ell+1})$$

$$\eta_{\ell+1/2} = (1/2)(\eta_\ell + y_{\ell+1}) \qquad \ell = 1, 2, 3$$

and evaluates DEN_ℓ and $DEN_{\ell+1/2}$ for $\ell = 1, 2, 3$.

Considering a side $(\ell, \ell+1)$, we introduce an auxiliary coordinate z so that $z = -1$, $z = 0$, and $z = +1$, respectively, at the points $\ell, \ell+(1/2)$ and $\ell+1$. Then one has an approximation for DEN along the side $(\ell, \ell+1)$

$$DEN(z) = DEN_{\ell+1/2} + a_1 z + a_2 z^2$$

with

$$a_1 = (DEN_{\ell+1} - DEN_\ell)/2$$

$$a_2 = (DEN_{\ell+1} - DEN_\ell - 2DEN_{\ell+1/2})/2$$

The extremum lies at

$$z = -\frac{a_1}{2a_2}$$

If $a_2 = 0$, it is practical to replace it by a small ...

10^{-3} say, to avoid an exceptional treatment ...
The extremum lies within the side of the tri...

$$|z| < 1.$$

It is given by

AD-A180 534 A DISCRETIZATION OF THE INTEGRAL EQUATION FOR THE TIME DEPENDENT LINEARIZ..(U) DAYTON UNIV OH RESEARCH INST K G GUDERLEY AUG 87 UDR-TR-87-26 AFWAL-TR-87-3061 2/2

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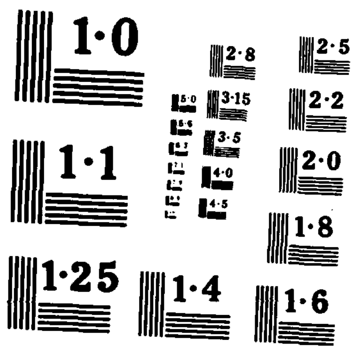
1·0
2·8
2·5
3·15
2·2
3·5
1·1
2·0
4·0
4·5
1·8
1·25
1·4
1·6

$$DEN_{ext,\ell,\ell+1} = \frac{-a_1^2}{4a_2}$$

If $|z| > 1$, we set $DEN_{ext,\ell,\ell+1} = DEN_\ell$. Then one has for the overall maximum and minimum

$$DEN_{max} = \max(DEN_\ell,\ DEN_{ext\ \ell,\ell+1};\ \ell = 1,\ 2,\ 3)$$

$$DEN_{min} = \min(DEN_\ell,\ DEN_{ext\ \ell,\ell+1};\ \ell = 1,\ 2,\ 3)$$

The ratio DEN_{max}/DEN_{min} is used as criterion for the choice of the integration formula. Of course, for triangles at some distance from the control point, one can dispense with this detailed evaluation.

SECTION XI

SPECIAL CHOICE OF THE TIME DEPENDENCE OF THE UPWASH

With the procedure described in the preceding sections, one can determine the function h and with it the potential ϕ and the pressures generated by an arbitrarily given wing displacement g(x,y,t).

In aeroelastic applications, the displacements of the wing surface are given by a moderately large number of known functions of x and y multiplied by unknown coefficients that depend upon t

$$g(t,x,y) = \sum_{1}^{n} a_n(t)g_n(x,y) \qquad (37)$$

In this section the procedures will be discussed for displacements expressed in this form. During this discussion, the underlying integro-differential equation is summarized in the form

$$\frac{d\vec{h}}{dt} + \int_{0}^{t} A(\bar{\tau})\vec{h}(t-\bar{\tau})d\bar{\tau} = \vec{w}(t) \qquad (38)$$

Setting $\bar{\tau} = t - \tau$, one obtains, alternatively.

$$\frac{d\vec{h}}{dt} + \int_{0}^{t} A(t-\tau)\vec{h}(\tau)d\tau = \vec{w}(t) \qquad (39)$$

This equation is closely related to Eq. (7). The elements of the vectors $\vec{h}$ and $\vec{w}$ are, respectively, the time dependent parameters by which h(t,x,y) (and with it the potential) and the time dependent values of w(t,x,y) at the control points are characterized. The matrix A has elements which depend (in Eq. 38) upon the retardation τ. It arises if one discretizes with respect to the space coordinates x and y, but considers τ as continuous. This was our original concept. In expressing the coefficient $C_{(ij),(mn)k}$, it had been assumed that the matrix

elements are given in tables with τ as independent variable and the necessary interpolations have been carried out. In contrast $A(\tau)$ is considered as a continuous function. One remembers that the points on the wing surface are denoted by pairs of subscripts. (This facilitates the identification of their neighbors.) If one thinks of a matrix in C, one has (i,j) as row and (m,n) as column indices. If one writes the equation in the above form, it is assumed that one first carries out the spatial discretization and afterwards performs the integrations over ξ and η required in B_7, B_8, or B_9 at the values of τ evaluated for the pertinent values of $(x-\xi)$ and $(y-\eta)$. In the procedures proposed in the first part of this report, the integration had been carried out over areas; the fact that within such an area, τ varies had been taken into account at the same time.

For displacements in the form Eq (37), the function w(t,x,y), which enters the aerodynamic part of the problem is given by

$$w(t,x,y) = \sum_1^n (\dot{a}_n(t)M^{-1}g_n(x,y) + a_n(t)\frac{\partial g_n(x,y)}{\partial x}) \qquad (40)$$

Here $a^{\cdot}$ denotes the derivative of a with respect to its argument. In the two dimensional case, one has, for instance, for a plunging motion

$$g(t,x) = a(t),$$

then

$$w(t,x) = M^{-1}a^{\cdot}(t)$$

For a rotating motion, one has

$$g(t,x) = a(t)x$$

then

$$w(t,x) = M^{-1}a^{\cdot}(t)x + a(t).$$

In the further discussions we write

$$w(t,x,y) = \sum_{\ell+1} c_{\ell}(t)\tilde{w}_{\ell}(x,y) \qquad (41)$$

In general, one function $g(x,y)$ will generate two functions $\tilde{w}_{\ell}$, namely $g(x,y)$, and $g_x(x,y)$. The vector $\vec{w}(t)$ is formed from the values of $w(t,x,y)$ at the control points

$$\vec{w}(t) = \sum_{\ell} c_{\ell}(t)\vec{w}_{\ell}$$

The terms of the sum are now considered separately.

For this purpose, the function c_{ℓ} is written in the form

$$c_{\ell}(t) = \int_{0}^{t+\alpha} c_{\ell}^{\cdot}(\sigma)H(t-\sigma)d\sigma + c_{\ell}(0)H(t), \qquad \alpha > 0 \qquad (42)$$

where the Heavyside step function $H(t-\tau)$ is defined by

$$\begin{aligned} H\ (t-\tau) &= 0 \qquad t - \tau < 0 \\ H\ (t-\tau) &= 1 \qquad t - \tau > 0 \end{aligned} \qquad (43)$$

The second term on the right of Eq. (42) can be omitted if one postulates $c_{\ell}(0) = 0$ and admits a discontinuity of c_{ℓ} at $t = 0$; $c_{\ell}^{\cdot}$ will then have a delta function at $\sigma = 0$. An alternative form of c_{ℓ} (not used here) is

$$c_{\ell}(t) = \int_{0}^{t+\alpha} c_{\ell}(\tau)\delta(t-\tau)d\tau, \qquad \alpha > 0$$

It is connected to Eq. (42) by an integration by parts. Notice that

$$\frac{\partial H}{\partial t} = -\frac{\partial H}{\partial \tau} = \delta(t-\tau) = \delta(\tau-t) \qquad (44)$$

Eq. (42) suggests that we write

$$\vec{h}_\ell(t) = \int_0^{t+\alpha} c_\ell^{\cdot}(\sigma)\tilde{h}_\ell(t,\sigma)d\sigma \tag{45}$$

Then one obtains from Eq. (39), by inspection,

$$\frac{d\tilde{h}_\ell(t,\sigma)}{dt} + \int_0^t A(t-\tau)\tilde{h}_\ell(\tau,\sigma)d\tau = \vec{w}_\ell \; H(t-\sigma) \tag{46}$$

The discussions which end with Eq. (52) show how Eq. (46) is obtained formally. One substitutes Eq. (45) and Eq. (42) into Eq. (39) and interchanges the sequence of integrations in the second term. Then the integration over σ is the outer integration in all terms, always with a factor $c_\ell^{\cdot}(\sigma)$. Since $c_\ell^{\cdot}(\sigma)$ is arbitrary, the other factors combined must vanish.

We set

$$t - \sigma = \bar{t} \tag{47}$$

$$\tau - \sigma = \bar{\tau}$$

and

$$\tilde{h}_\ell(t,\sigma) = \tilde{h}_\ell(\bar{t}+\sigma,\sigma) = \hat{h}_\ell(\bar{t},\sigma) \tag{48}$$

Then one obtains

$$\frac{d\hat{h}_\ell}{d\bar{t}}(\bar{t},\sigma) + \int_0^{\bar{t}} A(\bar{t}-\bar{\tau})\hat{h}_\ell(\bar{\tau},\sigma)d\bar{\tau} = \vec{w}_\ell H(\bar{t}) = \vec{w}_\ell, \quad \bar{t} > 0 \tag{49a}$$

$$\frac{d\hat{h}_\ell}{d\bar{t}}(t,\sigma) + \int_0^{\bar{t}} A(\bar{t}-\bar{\tau})\hat{h}_\ell(\bar{\tau},\sigma)d\bar{\tau} = 0, \qquad \bar{t} < 0 \qquad (49b)$$

From the initial condition

$$\tilde{h}_\ell(0,\sigma) = 0$$

which becomes

$$\hat{h}_\ell(-\sigma,\sigma) = 0$$

and Eq. (49b), one obtains

$$\hat{h}_\ell(0,\sigma) = 0 \qquad (50)$$

The parameter σ does not occur explicitly in Eq. (49a) and in initial condition (50). Therefore, $\hat{h}$ does not depend upon σ

$$\hat{h}_\ell(\bar{t},\sigma) = \hat{h}_\ell(\bar{t}) \qquad (51)$$

Eqs. (49a) and (50) are therefore rewritten

$$\frac{d\hat{h}_\ell}{d\bar{t}}(\bar{t}) + \int_0^{\bar{t}} A(\bar{t}-\bar{\tau})\hat{h}_\ell(\bar{\tau})d\bar{\tau} = \vec{w}_\ell \,, \qquad \hat{h}_\ell(0) = 0 \qquad (52)$$

This equation is solved within the desired region of values $\bar{t}$ for all vectors $\tilde{w}_\ell$, which occur in Eq. (41). Then

$$\hat{h}_\ell(\bar{t},\sigma) = \hat{h}_\ell(\bar{t}-\sigma) \qquad (53)$$

The operator on the left of Eq. (39) is denoted by L

$$L(\vec{h}) = \frac{\partial \vec{h}}{\partial t} + \int_0^t A(t-\tau)\vec{h}(\tau)dt$$

Eq. (46) written in the form

$$L(\tilde{h}_\ell(t) = c_\ell(t)\vec{w}_\ell$$

is solved by

$$\tilde{h}_\ell(t) = L^{-1}(c_\ell(t)\vec{w}_\ell = \int_0^t c_\ell^{\cdot}(\sigma)\hat{h}_\ell(t-\sigma)d\sigma \qquad (54)$$

(provided that $c_\ell(0) = 0$). As was mentioned above, because of this condition one must allow that in general $c_\ell^{\cdot}(\sigma)$ has a delta function at $\sigma=0$.

Applying an integration by part to the right hand side of the last equation, one obtains

$$\tilde{h}_\ell(t) = L^{-1}(c_\ell(t)\vec{w}_\ell) = \int_0^t c_\ell(\sigma)\hat{h}_\ell^{\cdot}(t-\sigma)d\sigma \qquad (55)$$

(because $c_\ell(\sigma)$ vanishes from $\sigma = 0$ and $\hat{h}_\ell$ vanishes for $\sigma = t$). The function $\hat{h}_\ell^{\cdot}$ arises as one integrates Eq. (52). Then

$$\vec{h}(t) = L^{-1}(\vec{w}(t) = \sum_\ell \tilde{h}_\ell(t) = \sum_\ell \int_0^t c_\ell(\sigma)\hat{h}_\ell^{\cdot}(t-\sigma)d\sigma \qquad (56)$$

The potential at the control points, again written to form a vector, is then given by

$$\vec{\phi}(t) = 2\pi\vec{h}(t) \qquad (57)$$

The pressures, in dimensionless form, are given by

$$p(x,y,t) = \mp(M^{-1}\phi_t + \phi_x) \qquad (58a)$$

where the upper and lower signs refer, respectively, to the upper and lower wing surface. It follows from Eqs. (52) and (50) that

$$\hat{h}_{\ell}(0) = \vec{w}_{\ell} \tag{58b}$$

The pressures enter into the equations of motions for the wing in the form

$$I = \iint p(t,x,y)q_n(x,y)dxdy \tag{59}$$

The functions q_n are identical with, or at least closely related to the function $g_n(x,y)$ of Eq. (37). They are usually available in the analytic form so that $\partial g_n/\partial_x$ is readily formed. Using Eq. (58a), one then obtains for these integrals I

$$I = M^{-1} \iint \phi_t q_n(x,y)dxdy + \iint \phi_x q_n(x,y)dxdy$$

In the second term an integration by parts with respect to x is carried out (because $(\partial q_n/\partial x$ is more readily available than ϕ_x). Taking into account that $\phi = 0$ at the leading edge, one obtains

$$I = M^{-1} \iint \phi_t q_n(x,y)dxdy + \int \phi_{\mathrm{trailing}}\, q_n(x_{\mathrm{trailing}}(y),y)dy$$
$$- \iint \phi(x,y)q_{n,x}(x,y)dxdy \tag{60}$$

Assume that functions u(x,y) and q(x,y) are characterized by their values at pivotal points, and thus result in vectors $\vec{u}$ and $\vec{q}$. Then we write

$$\iint_{\mathrm{wing}} u(x,y)q(x,y)dxdy = [\vec{u}, B\vec{q}]$$

The right side is considered as a scalar product. The matrix B arises in the integration process (loosely speaking, it is the substitute for the area element).

With this notation we rewrite Eq. (59) cast in the form (60). The potential ϕ is expressed in terms of h by Eq. (57) and h is expressed by Eq. (56); $\hat{h}^{\cdot}(0)$, which arises when one forms $h^{\cdot}(t)$ from Eq. (56) by differentiating with respect to the upper limit of the integral is found in Eq. (58b). Then

$$\iint\limits_{\text{wing}} p_\ell(t,x,y)q_n(x,y)dxdy = \mp 2\pi\{M^{-1}(c_\ell(t)[\vec{q}_n, B\vec{w}_\ell] + \int_0^t c_\ell(\sigma)[\vec{q}_n, B\hat{h}^{\cdot\cdot}_\ell(t-\sigma)]d\sigma)$$

$$+ \int\limits_{\text{trailing edge}} h_{\text{trailing}',\ell q_n}(x_{\text{trailing}}(y),y)dy$$

$$- \int_0^t c_\ell(\sigma)[\vec{q}_{n,x} B\hat{h}_\ell(t-\sigma)d\sigma)]d\sigma\}$$

The expression $[q_n, B\hat{h}^{\cdot\cdot}_\ell]$ and $[q_{nx}, B\hat{h}^{\cdot}_\ell]$ can be evaluated in conjunction with the evaluation of $\hat{h}_\ell$. If one does not need to examine the pressure distribution by itself, this will reduce the storage requirements considerably.

SECTION XII
TRANSITION TO A SMALLER VECTOR $\vec{h}$

Because of the size of the matrix A (in Eq. 39), the integration of the integro-differential equation for $\vec{h}$ is probably best carried out by a predictor-corrector method, rather than by some implicit procedure. (The vector $\vec{h}$ may have several hundred components.) An explicit method, like a predictor corrector method, becomes unstable if the time step is too large. The limit of the time step which arises in this manner is dependent upon the discretization in space. Incidentally, according to the experiences gained for partial differential equations without memory (as contrasted to the present problem where memory terms are introduced by the retardation), the approximation to the original problem (that is the problem before the discretization) is poorer if one uses a time step which is smaller than the maximum time step compatible with stability. In the first stages of the response to a function w whose time dependence is given by a step function, this restriction to a small time step is no disadvantage because a large time step which is possible in implict methods will average out the rapid changes of the potential characteristic of these initial stages.

After this initial phase, the function $\vec{h}(t)$ changes more slowly and finally approaches the values pertaining to the steady state generated by the function $w_\ell(x,y)$ (here $\vec{w}_\ell$). It is then possible to proceed in larger time steps provided that $\vec{h}$ is redefined.

Beside the required time resolution, the grid size is determined by the accuracy desired for the integrations over ξ and η. The original size of the grid will probably be determined by the accuracy desired in the initial time response; the grid so obtained is likely to be sufficient for the integration with respect to ξ and η. The results obtained for this fine grid are the basis for the transition to a representation of $\vec{h}$ by a reduced number of parameters.

In this process, one can distinguish between three steps. In the first step one represents $\vec{h}$ by a reduced number of parameters. This leaves the number of points at which the upwash is evaluated unchanged, and with it the number of conditions that can be imposed to this form of $\vec{h}$. In the second step, one reduces the number of conditions derived from the integral equation, so that it matches the number of parameters used to characterize $\vec{h}$. In the third step, one enlarges the time interval.

We assume immediately that the time dependence of $\vec{w}$ is given by a step function. Then the function $\vec{h}$ approaches a steady state and changes more and more slowly after an initial phase. We consider the equation obtained after a discretization in space but without discretization in time

$$\frac{\partial \vec{h}}{\partial t} + \int_0^t A(\bar{\tau})\vec{h}(t-\bar{\tau})d\bar{\tau} = \vec{w}$$

where $\vec{w}$ is constant for t positive. Assume that the initial phase of the time evolution of $\vec{h}$ during which this equation is used in its original form terminates at time t_1. Then we write

$$\frac{d\vec{h}}{dt} + \int_0^{t-t_1} A(\bar{\tau})\bar{h}(t-\bar{\tau})d\bar{\tau} + \int_{t-t_1}^{t} A(\bar{\tau})\bar{h}(t-\bar{\tau})d\bar{\tau} = \vec{w}$$

In the second integral, the argument of $\vec{h}$ varies from t_1 at the lower limit to 0 at the upper limit. For these values of t, $\vec{h}$ is already known. Presumably, it changes fairly rapidly in space as well as in time. Therefore, one cannot economize in the evaluation of this integral. One will remember, however, that the elements of the matrix $\vec{A}$ are zero if the retardation τ exceeds a certain value (which depends upon the dimensions of the wing). Exceptions are matrix elements which give the influence of the trailing edge on the upwash at other parts of the wing,

because the trailing edge data incorporates information about the wake, which theoretically, at least, extends to infinity.

The transition to fewer parameters in the representation of $\vec{h}$ can, however, be made in the first integral of the last equation. With suitably chosen functions $f_\ell^{(1)}(x,y)$, one sets

$$h(t,x,y) = \sum_{\ell=1}^{\ell_1} a_\ell(t) f_\ell^{(1)}(x,y)$$

By making the transition from $h(t,x,y)$ to $\vec{h}(t)$, one obtains

$$\vec{h}(t) = M^{(1)}\vec{a}(t)$$

where the matrix $M^{(1)}$ has as column vectors the functions $f_\ell^{(1)}$ evaluated at the control points

$$M^{(1)} = \left[f_1^{(1)}(x_i,y_i) \;\vdots\;\vdots\; f_\ell^{(1)}(x_i,y_i) \;\vdots\;\vdots \right]$$

and (61)

$$\vec{a}(t) = \begin{bmatrix} a_1(t) \\ \vdots \\ a_\ell(t) \\ \vdots \end{bmatrix}$$

Here i is the subscript for a control point in a one-dimensional numbering system (and x_i,y_i are its coordinates). One thus arrives at the equation

$$M^{(1)}\frac{d\vec{a}}{dt} + \int_0^{t-t_1} [A(\bar{\tau})M^{(1)}]\vec{a}(t-\bar{\tau})d\bar{\tau}) + \int_{t-t_1}^{t} A(\bar{\tau})\vec{h}(t-\bar{\tau})d\bar{\tau} = \vec{w} \qquad (62)$$

With this equation, one can evaluate the upwash at all control points. The new dependent variable is the vector $\vec{a}$. The vector

$\vec{a}$ is considerably smaller than the vector $\vec{h}$. Of course, this equation cannot be satisfied exactly at all control points.

In a second step, one premultiplies the last equation by a matrix $M^{(2)}$ with ℓ_1 rows and a number of columns which matches the dimension of $\vec{h}$. One then obtains

$$
\begin{aligned}
M^{(3)}\frac{d\vec{a}}{dt} &+ \int_{0}^{t-t_1} M^{(4)}(\tau)\vec{a}_n(t-\bar{\tau})d\bar{\tau} \\
&+ \int_{t-t_1}^{t} M^{(5)}\,\vec{h}\,(t-\bar{\tau})d\bar{\tau} = \vec{w}^{(1)}
\end{aligned}
\tag{63}
$$

where

$$M^{(3)} = M^{(2)}M^{(1)}$$

$$M^{(4)}_{(\tau)} = M^{(2)}A(\bar{\tau})M^{(1)}$$

$$M^{(5)}_{(\tau)} = M^{(2)}A(\bar{\tau})$$

$$\vec{w}^{(1)} = M^{(2)}\vec{w}$$

The matrix $M^{(2)}$ may single out a suitable number of control points, or its rows may be given by the values of ℓ_1 weight functions at the control points. In the last form, the equation holds for any value of $t > t_1$. In the description of the program we assumed, however, that one deals with equidistant values of t and that the interpolations needed are incorporated into the matrix $C_{(ij),(mn),k}$. There the numbering of the control points by pairs of subscripts is used; for the rows (ij), for the columns (m,n). Introducing an analogous notation in the matrices $M^{(1)}$ and $M^{(2)}$, one has for the matrix elements

$$M^{(1)}_{(mn),\ell} \quad f_{\ell}^{(1)}(x_{mn},y_{mn})$$

$$M^{(2)}_{\ell,(ij)} = f^{(2)}_{\ell}(x_{ij}, y_{ij})$$

Then, one has

$$M^{(3)}_{\ell_1 \ell_2} = \sum_{i,j} M^{(2)}_{\ell_1 (ij)} M^{(1)}_{(ij)\ell_2}$$

$$M^{(4)}_{\ell_1 \ell_2; k} = \sum_{i,j} \sum^{(2)}_{num} M^{(2)}_{\ell_1 (ij)} C_{(ij)(mn);k)} M^{(1)}_{(mn),\ell_2}$$

The Matrix $M^{(4)}$ which previously depended upon $\bar{\tau}$, now depends upon a parameter k. This is expressed by the notation $M^{(4)}(k)$. Similarly for $M^{(5)}$. There elements are given by

$$M^{(5)}_{\ell_1,(mn);k} = \sum_{i,j} M^{(2)}_{\ell_1 (ij)} C_{(ij)(mn),k)}$$

$$w^{(1)}_{\ell} = \sum_{i,j} M^{(2)}_{\ell_1,(ij)} w_{(ij)}$$

Eq. (44) now appears in the form

$$M^{(3)} \left(\frac{d\vec{a}}{dt}\right)_k + \sum_{\ell=0}^{k-k_1-1)} M^{(4)}_{\ell} \vec{a}_{k-\ell} + \sum_{\ell=k-k_1}^{k} M^{(1)}_{\ell} \vec{h}_{k-\ell} = \vec{w}^{(1)}; \ k > k_1$$

The derivative $\left(\frac{d\vec{a}}{dt}\right)_k$ is obtained by premultiplication with $M^{(3)-1}$:

$$\left(\frac{d\vec{a}}{dt}\right)_k + \sum_{\ell=0}^{k-k_1-1)} M^{(6)}_{\ell} \vec{a}_{k-\ell} + \sum_{\ell=k-k_1}^{k} M^{(7)}_{\ell} \vec{h}_{k-\ell} = \bar{w}^{(2)}$$

with

$$M^{(6)}_{\ell} = M^{(3)-1} M^{(4)}_{\ell}$$

$$M^{7}_{\ell} = M^{(3)-1} M^{(5)}_{\ell}$$

$$\vec{w}^{(2)} = M^{(3)-1}\vec{w}^{(1)}$$

In addition, one needs initial conditions for the vector $\vec{a}$ at $k = k_1$. Known from the preceding computation is $\vec{h}$ for $k = k_1$. One must approximate $\vec{h}(k_1)$ by $M^{(1)}a(k_1)$

This is the formulation for the original time step. A reduction in the number of parameters describing $h(t,x,y)$ is likely to tolerate a larger time step. Such a transition can be made if the new time step is a multiple of the original one. One computes the vector $\vec{a}$ only for values belonging to the larger time step and expresses intermediate values of the vector $\vec{a}$ by linear interpolation. We show the procedure for tripling the time step. (It might be more realistic only to double the time step. Tripling is chosen to indicate how one proceeds in general.) Then only values for $k = k_1 + 3s; s = 0, 1...$ occur. Let

$$\vec{a}_{k_1+3s} = \vec{a}_s.$$

In the second sum of the last equation, the summation extends

for $s = 0$, $(k=k_1)$, from $\ell = 0$ to $\ell = k_1$

for $s = 1$, $(k=k_1+3)$, from $\ell = 3$ to $\ell = k_1+3$

for $s = 2$, $(k = k_1+6)$ from $\ell = 6$ to $\ell = k_1+6$

In all cases the subscript of $k-\ell$ of $\vec{h}$ varies from 0 to k_1. The number of terms in the sum is always the same and does not depend upon the size of the new integral. However, for s sufficiently large, all elements of the matrix $M^{(5)}$ vanish except for those pertaining to values of $\vec{h}$ at the trailing edge.

In the first sum, the interpolation for the vectors $\vec{a}_{k-\ell}$ is introduced. One has

$$\vec{a}_{k_1+3s} = \hat{a}_s$$

$$\vec{a}_{k_1+3s+1} = (2/3)\hat{a}_s + (1/3)\hat{a}_{s+1}$$

$$\vec{a}_{k_1+3s+2} = (1/3)\hat{a}_s + (2/3)\hat{a}_{s+1}$$

The summand of the first sum are now ordered by the subscript of $\hat{\vec{a}}$. For $(k = k_1+3)$, one obtains three terms ($\ell=0$, $\ell=1$, $\ell=2$)

$$M_0^{(6)}\hat{a}_1 + M_1^{(6)}((2/3)\hat{a}_1 + (1/3)\hat{a}_0) + M_2^{(6)}((1/3)\hat{a}_1 + (2/3)\hat{a}_0)$$

$$= (M_0^{(6)} + (2/3)M_1^{(6)} + (1/3)M_2^{(6)})\hat{a}_1 + ((1/3)M_1^{(6)} + (2/3)M_2^{(6)})\hat{a}_0$$

For $s = 2$ $(k = k_1 + 6)$, one obtains

$$M_0^{(6)} + (2/3)M_1^{(6)} + (1/3)M_2^{(6)})\hat{a}_2$$

$$+ ((1/3)M_1^{(6)} + (2/3)M_2^{(6)} + M_3^{(6)} + (2/3)M_4^{(6)} + (1/3)M_5^{(6)})\hat{a}_1$$

$$+ ((1/3)M_4^{(6)} + (2/3)M_5^{(6)})\hat{a}_0$$

For general s $(k = k_1 + 3_s)$, one obtains

$$\hat{M}_0^{(6)}\hat{a}_s + \hat{M}_1^{(6)}\hat{a}_{s-1} + \ldots \hat{M}_n^{(6)}\hat{a}_{s-n} \ldots M^{(6,s)}\hat{a}_0 \qquad (64)$$

with

$$\hat{M}_0^{(6)} = M_0^{(6)} + (2/3)M_1^{(6)} + (1/3\ M_2^{(6)}$$

$$\hat{M}_n^{(6)} = (1/3)M_{3n-2}^{(6)} + (2/3)M_{3n-1}^{(6)} + M_{3n}^{(6)} + (2/3)M_{3n+1}^{(6)} + (1/3)M_{3n+2};$$

$$n = 1, 2, \ldots$$

These quantities are independent of s in contrast to

$$\hat{M}^{(6,s)} = (1/3)M^{(6)}_{3s-2} + (2/3)M^{(6)}_{3s-1}$$

The first sum then becomes

$$\sum_{n=0}^{s-1} \hat{M}^{(6)}_{n} \hat{a}_{s-n} + \hat{M}^{(6,s)}_{s}\hat{a}_{0}$$

and one obtains

$$\frac{d\hat{a}}{dt}_{s} + \sum_{n=0}^{s-1} \hat{M}^{(6)}_{n}\hat{a}_{s-n} + \hat{M}^{(6,s)}\hat{a}_{0}$$
$$+ \sum_{\ell=3s}^{k_0+3s-1} M^{(7)}_{\ell} \; h_{k_0+3s-\ell} = \vec{w}_{(2)} \tag{65}$$

The matrices M and the vectors $\vec{w}^{(1)}$ and $\vec{w}^{(2)}$ can be evaluated in advance. For s = 0, the first sum is zero.

SECTION XIII

EXAMPLES FOR THE TRANSITION TO A SMALLER VECTOR $\vec{h}$

The procedure is illustrated by two examples for the two-dimensional case. With some modifications the ideas can be carried over to the three-dimensional case.

We assume that control points have been chosen along the chord of the profile at 10 equal intervals. (One may find it desirable to use more control points, but 10 is sufficient to illustrate the procedure.) In the first example, the size of the interval is doubled, the amount of labor to proceed over an equal time span is then reduced by a factor of 1/4.

The function h is now expressed by its value of only five control points. For the value of h at the control points that have been left out are determined by linear interpolation between the neighbors. At the leading edge, one has a square roots singularity. This is taken into account when h_1(h at control point 1) is expressed by the value of h_2. One then has

$$\begin{bmatrix} h_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ h_2 \end{bmatrix} = M^{(1)} \begin{bmatrix} a_1 \\ \cdot \\ \cdot \\ a_5 \end{bmatrix}$$

where

$$M^{(1)} = \begin{bmatrix} \sqrt{1/2} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The condition for the vector $\vec{a}$ are imposed at the new control points. Accordingly

$$M^{(2)} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then

$$M^{(2)}M^{(1)} = M^{(3)} = I_5$$

where I_5 is the 5 by 5 identify matrix.

One then evaluates $M^{(4)} = M^{(6)}$ and $M^{(5)} = M^{(7)}$. For the two-dimensional case, the rows and columns of the matrices C are numbered by single subscripts.

The integration with respect to t can now be carried out with twice the original time interval

$$k = k_1 + 2s.$$

Then, one has the following formulae

$$\hat{M}_0^{(6)} = M_0^{(6)} + (1/2)M_1^6$$

$$\hat{M}_n^{(6)} = (1/2)M_{2n-1}^{(6)} + M_{2n}^{(6)} + (1/2)M_{2n+1}^{(6)}$$

$$\hat{M}^{(6,S)} = (1/2)M_{2s-1}^{(6)}$$

The initial conditions for $\vec{a}$ are the value of $h(k_1)$ at the new control points

$$(\hat{a}_n)_o = (h_{2n})_{k_1}$$

The problem then appears in the form as shown in Eq. (65)

$$\frac{d\hat{a}}{dt}s + \int_{n=0}^{s-1} \hat{M}_n^{(6)}\hat{a}_{s-n} + \hat{M}^{(6,s)}\hat{a}_o + \sum_{\ell=2s}^{k_o+2s-1} M_\ell^{(7)} h_{k_o+2s-\ell} = \vec{w}(2)$$

Further details are left the the reader.

This procedure is somewhat crude insofar as it expresses h at the intermediate control point by linear interpolation between the control points except for the interval next to the leading edge. While the approximation for h by a piecewise linear function in 10 intervals may be sufficient, it may not be sufficiently accurate if one has only 5 intervals. The approach is sensible if the original interval had been chosen because of the desire to obtain a close time resolution while it is unnecessarily fine for the representation of the final steady state. In the two-dimensional case, the labor required to proceed by the same time is reduced by a factor of 1/4; in the three-dimensional case by a factor of 1/8. One can, of course, introduce a matrix $M^{(1)}$ in which one uses higher order interpolation formulae for h. The next procedure is of this character insofar as uses for the $f^{(1)}$ analytical expressions applied over the whole profile.

For this purpose, one can use functions derived from linearized airfoil theory. In the three-dimensional case, one

may use such expression for the chordwise dependence of h, their coefficients are then functions of a spanwise coordinate.

Classical two-dimensional airfoil theory is partially summarized in Appendix E of Reference 1. The profile is represented by a slit in the x,y plane extending along the x axis from -1 to +1. The x,y plane with this slit is mapped conformally into the exterior of a circle with radius 1. At this circle, the potential (in this case the function h), antisymmetric with respect to the x axis, is expressed by a complete system of functions

$$h_n = -\sin \theta$$

where θ is the polar angle. For the mapping from the circle to the slit in the x,y plane one has

$$x = \cos \theta \ ; \ (y=0)$$

Thus, a system of functions suitable to represent h (without allowing for circulation) is given by

$$\hat{h}_n = -\sin (n\theta)$$

with

$$\sin \theta = (1-x^2)^{1/2}$$

To these expressions of h, one adds a circulation term so that for $\theta = 0$, which is the map of the trailing edge $dh/d\theta = 0$. (One remembers that $dh/d\theta$ in the circle plane is the tangential derivative of h along the circle.) This condition ensures that in the x,y plane the velocity at the trailing edge is finite. Thus

$$h_n = -\sin(n\theta) + n(\theta-\pi)$$

The upwash pertaining to such a function h_n at the slit in the x,y plane is found according to Appendix E of Reference 1 as

$$w_n = n \sin(n\theta)/\sin\theta$$

The upwash generated by the circulation term is zero. The expression w_n is a polynomial in x. If the function w prescribed in the program for the unsteady flow is such a function w_n and its time dependence is given by a step function, then the asymptotic response will be the function h_n. One has the orthogonality relation

$$\int_0^\pi \sin(n\theta)\,\sin(n\theta)d\theta = 0; \quad m \neq n$$

hence

$$\int_0^\pi w_n\, w_m \sin^2\theta d\sigma = 0, \quad m \neq n$$

But

$$\cos\theta = x$$

$$-\sin\theta d\sigma = dx$$

Hence

$$\int_{-1}^{+1} w_n\, w_m\, (1-x^2)^{1/2}dx = 0, \quad m \neq n$$

By this formula one can decompose an upwash given by

$$w = \sum_1^n a_n w_n$$

into the individual components, which are generated by the functions h_n.

During the transition to the asymptotic value, the wake is not fully developed. Therefore, the functions h_n by themselves are not quite sufficient to approximate the potential. The individual expressions give at the trailing edge

$$h_n(x) = \text{const} + (x-x_{trail})^{3/2}$$

therefore, $h_{n,x} = 0$. But at the trailing edge, one must prescribe the conditions that no pressure jump occurs between the upper and lower side of the profile. Therefore,

$$\phi_x + M^{-1}\phi_t = 0$$

for the trailing edge. To allow for this expression, we include in the list of functions used for the representation of h, a function

$$\bar{w}_{N+1} = (x+1)^{1/2}(x-1)$$

At the trailing edge it gives a zero contribution to h and a value of $2^{1/2}$ for the value of h_x. At the leading edge, it has as all other functions a square root singularity. This is then the set of functions $f^{(1)}$ used to express h in terms of the parameters a_n.

A matrix $M^{(2)}$ chosen in such a manner that in the steady case it gives the above decomposition of the downwash, is likely to be pratical. Therefore

$$f_n^{(2)} = w_n(1-x^2)^{1/2} = \sin(n\theta); \qquad n=1,\ldots,N$$

This defines N of the functions $f^{(2)}$ which appear in the row vectors of the matrix $M^{(2)}$. One needs an additional condition, for this we choose the requirement that the original discretized

expression for the upwash is satisfied at the trailing edge. The trailing edge is singled out, because of its unique role in determining the vortex shedding in the wake.

To carry out the procedure for the new representation for h, one must have the initial conditions for the vector $\vec{a}_{k1}$. It is assumed that the procedure in its original form has been carried out up to a station k_1 in time so that the vector h_{k1} is known. Then one must find the values of a_n which give an acceptable approximation for $\vec{h}_{k1}$

$$\vec{h}_{k_1} = \sum_1^n a_{n,o} h_m + a_{N+1}(x+1)^{1/2}(x-1)$$

From the known values of h_{k1}, one determines numerically h_x at the trailing edge. Then, because $h_{m,x}=0$

$$a_{N+1,0} = 2^{1/2} h_{x\ \text{trailing edge}}$$

Now we are left with

$$\sum_1^N a_{no}\vec{h}_n = \vec{h}_{k1} - a_{N+1}(x+1)^{1/2}(x-1)$$

where one writes the last expression as a vector with components obtained by inserting the values of x for the original points.

There are orthogonality relations for the functions h_w if one leaves out the contribution of the circulation. The combined contribution of the circulation at the trailing edge gives $\vec{h}_{k1}$ at the trailing edge. Therefore

$$\sum_1^N a_{no}\vec{h}_n = \vec{h}_{k1} - a_{N+1}(x+1)^{1/2}(x-1) + \vec{h}_{k1,\text{trailing edge}}\left(\frac{\theta-\pi}{\pi}\right)$$

where the values of θ are re-evaluated at the control points.

The orthogonality conditions for the $\vec{h}_n$ are

$$\int_0^1 \sin\theta \sin(m\theta)d\theta = 0$$

or

$$\int_{-1}^{+1} \sin\theta \sin(m\theta)\ (1-x^2)^{1/2}dx = 0$$

$\sin(m\theta)/\sin\theta$ is a polynomial in x. Since $\vec{h}_{k1}$ is given only at the control points, one must use some simple approximate integration formula if one applies the orthogonality condition. The matrix then obtained for computing the coefficients a_{no} is nearly diagonal. Therefore, it is rather simple to obtain the values of a_{no}. If one uses these values of a_{no} and evaluates the potential at the trailing edge given by

$$-\sum_1^N a_{no} n\pi$$

one will, in general, not obtain the original value of $\vec{h}_{k1}$ at the trailing edge. It may be desirable to have continuity for the potential at the trailing edge between the values computed by the original procedure and that computed with the vectors $\vec{a}$. Then one must take some correction to the values of a_{no}.

With this procedure, it may be possible to carry out a more radical reduction in the size of the problem. And, it may also be possible to apply a much larger time step.

Other points of view than those anticipated here, may, of course, emerge in the course of numerical experimentation.

SECTION XIV

THE METHOD IN THE FRAMEWORK OF AN AEROELASTIC PROBLEM

The motivation for the development of the method described here has been the treatment of unsteady aeroelastic problems in the physical space rather than in the frequency space. This determines the manner in which the results will be used and the required accuracy. We discuss this in some detail.

The basic equations of aeroelasticity express the equilibrium between the elastic forces, the damping forces (both within the structure), the acceleration forces (d'Alembert forces), and the aeroelastic forces. All these forces are expressed in terms of the displacements of the wing, in this report denoted by $g(t,x,y)$. The elastic forces are expressed in terms of the g itself, the damping forces by $\partial g/\partial t$, the acceleration forces by $\partial^2 g/\partial t^2$, all expressions taken at the current time, the aerodynamic forces are expressed by $\partial g/\partial t$ and g at the current and at retarded times. One can consider $g(t,x,y)$ at fixed time t as the element of a function space, called displacement space; $\partial g/\partial t$ and $\partial^2 g/\partial t^2$ are then other elements of this space. If t varies these elements will, of course, move along curves in the displacement space. The elastic, damping, mass, and aerodynamic forces are obtained from these elements by operators, which map these elements from the displacement space into a "force" space.

The equations of motions express the requirement that the sum of the forces generated by these mappings give at all times the zero element of the force space.

If one restricts in the process of discretization the element g to a finite dimensional subspace of the displacement space, then the above operators will map the elements of this subspace into subspaces of the force space. These subspaces are of the same dimension as the original subspace of the displacement space, but, in general, they are not the same.

Accordingly the sum of these forces will not vanish perfectly. An exception occurs, if one restricts oneself to the mass and stiffness operators. The mass and stiffness operator operating on functions of x and y define an eigenvalue problem. For the eigensolutions the mapping due to the mass and the stiffness operator give, except for a factor of proportionality (the eigenvalue), the same vector in force space. If the subspace used for the approximation of the deformations is spanned by a finite number of eigenvectors then the subspaces obtained by applying the mass and stiffness operator are identical. By defining the spaces in a somewhat different manner and admitting complex elements one can also include the damping operator. The aerodynamic operator, however, is different for each value of the retardation and cannot be included in such an approach. If one had only a stiffness and a mass operator and used a subspace spanned by eigenfunctions, then one would satisfy the differential equations of the homogeneous problem perfectly (an approximation arises only in the manner in which one satisfies the initial conditions).

As mentioned before, the aerodynamic forces are determined by operators which are different for each value of the retardation, and therefore do not lend themselves to such an approach. If one applies a discretization in displacement space, then one cannot satisfy the differential equations of motions (which are relations in force space) perfectly.

The problem is ultimately reduced to a system of ordinary differential equations for the coefficients of the spanning vectors of the deformation space. To obtain such a system, one first expresses the displacement vector (the elements of the displacement subspace) by the sum of spanning vectors of the displacement space multiplied by time depend coefficient (so far unknown) and substitutes them into the equations of motion (in other words one applies to them the stiffness, damping, mass, and aerodynamic operators). This gives a vector in the force space. Subsequently, one multiplies the equations of motion by weight

functions and then integrates over x and y. This generates a system ordinary differential equations for the coefficients of the spanning vectors. In problems where the concept of eigenvectors is applicable there is no question about the choice of the weight functions, they are the spanning vectors of the displacement space. Because of the orthogonally relations which exist in this case, one even obtains single equations for the individual coefficients. The simple example of an oscillating string with non-uniform mass distribution can serve as an illustration for this situation.

Such thorough simplifications are not feasible in the presence of aerodynamic forces. The eigenfunctions of the problem without aerodynamic forces lose their physical and mathematical significance, although one can still expect that the eigenfunctions pertaining to the lowest frequencies are a "reasonable" spanning of the displacement space. Instead of determining these eigenfunctions, one describes the displacements by a finite number of linearly independent reasonably smooth functions. If one chooses, for instance 10 such functions, then it is quite likely that a linear combination of them will give good approximations to the first few eigenfunctions, and this is sufficient. If, in a specific situation, the coefficients ultimately obtained for those functions which have the greatest waviness are small, then the choice of the system of spanning functions is "reasonable."

The weight functions applied to the force equations in order to arrive at a system of ordinary differential equations for the coefficients are chosen in analogy to the treatment in terms of eigenfunctions, i.e., one chooses the functions which span the displacement subspace. By the subsequent integration over x and y one obtains the stiffness, damping, mass and the aerodynamic matrices. The latter depend upon the retardations. The others are time independent. The weight functions applied here are the same as the functions q mentioned in Section XI. If the stiffness damping and mass matrix are determined

experimentally, then one must, of course, use in the processing of the aerodynamic forces the weight functions of the experimental arrangement. There is nothing new in this description, every flutter computation is based on this idea.

The model obtained after one has introduced finite stiffness damping and mass matrices defines certain eigenfrequencies and by them a measure for the time resolution which is of technical interest. This, in turn, defines the maximum grid size by which one achieves the desired time resolution. (Another criterion for the upper bound of the grid size is the accuracy of the integrations which appear in the determination of the aerodynamic force.)

To illustrate the relation between time interval and frequency we consider

$$\ddot{a} + \nu^2 a = f(t)$$

This equation can be interpreted as the equation for the coefficient of one eigenfunction under the influence of a forcing function f(t); which takes the place of the aerodynamic forces.

The solution (with initial conditions $a = 0$, $\dot{a} = 0$) is given by

$$a(t) = \nu^{-1}\{\sin(\nu t)\int_0^t f(\tau)\cos(\nu\tau)dt - \cos(\nu t)\int_0^t f(\tau)\sin(\nu t)dt\}$$

If f(t) is of significant size only in the internal $0 < t < t_1$ $t_1 \ll \nu^{-1}$, then one obtains as approximation for $t > t_1$

$$a(t) \sim \nu^{-1}\sin(\nu t)\int_0^t f(\tau)dt$$

In other words, it does not matter, if in an interval $\Delta t \ll \nu^{-1}$ the function f(t) is known only in the average. One might consider $\Delta t = \nu^{-1}\, 10^{-1}$ as sufficient.

The aerodynamic forces are governed by two phenomena which may have different time scales. One is the propagations of waves over the wing. We have considered a function w(t,x,y) whose time dependence is given by a step function. At the time immediately after the step, the pressure at the wing is solely determined by the local value of w(x,y). But this pressure distribution is not compatible with the pressures outside of the wing (which are zero). The pressure adaptation therefore needed, occurs by waves which travel over the wing. In the two dimensional case one has only waves traveling in upstream and downstream directions. The downstream travel occurs at higher velocities than the upstream travel, especially at high subsonic Mach numbers. In the three-dimensional case, one has waves in all directions. These waves are reflected at the wing contour, but they attenuate with distance because the underlying phenomena space are three dimensional.

A second time scale is given by the development of the wake. While the adaptation to the outside pressures is practically over after a certain (possibly rather short) time, the wake keeps extending and changing. Especially at low Mach numbers the development of a steady wake will take a considerable time. The wake vorticity affects the upwash on the wing and with it the pressure distribution. Some time after the step in the upwash, the upwash at the wing generated by the wake will change only slowly and the adaptation to this upwash will probably occur in a quasisteady manner. In testing the method one should be aware of this distinction and try to recognize the relative importance of these phenomena. The characteristic times for these phenomena should be considered in relation to the time resolution imposed by mechanical properties of the system. The transition from a finer to a coarser grid described in the preceding section is the practical consequence of the difference between the initial phase where quickly moving waves determine the pressure distribution and a later phase where such waves play a minor role and the phenomena are quasisteady.

SECTION XV

EFFECT OF A TRUNCATION OF THE WAKE

The wake generated by an upwash $w(t,x,y)$ whose time dependence is given by a step function, starts with length zero. The end of the wake moves downstream with the velocity of the particles, that is with the free stream velocity. If one follows the evolution of the flow field for a long time then the wake will become very long and the labor to take the distant parts into account will be excessive, while their influence on the upwash at the wing will become smaller and smaller. In practice the wake will be terminated after a finite distance.

We try here to describe the effect of such a truncation in general terms. Actually this is done only for the final steady state, which arises in response to a step function in time, for the interpretation will be based on the concept of vortices in a steady flow.

For the two-dimensional case h_y is zero by definition. In the functions B_7 and B_8 one encounters factors $h^{(2)}$ and $h^{(3)}$ (in a more familiar notation h_x and h_y). Disregarding distant parts of the wake means that one sets h_x and h_y equal to zero. In a wake there is no pressure difference between the upper and lower sides in a steady linearized flow ϕ_x and with it h_x are zero. By truncating the wake, one sets h_x equal to zero prematurely, but in the final stage h_x is zero in any case. Therefore, the final steady state for a two-dimensional flow remain unaffected by the truncation of the wake.

In the three-dimensional steady case one has vortices trailing along the stream lines (here lines in the direction to the x axis) out to infinity. Therefore, $h_y \neq 0$ even in the steady case. h_y is the local vorticity strength in the wake, the vorticity vector has the x direction. But every vector field $\vec{u}$ satisfies div curl $\vec{u} = 0$. Vorticity lines cannot end somewhere in the interior of a vector field. It seems as if this

requirement is violated, if one disregards the values of ϕ_y in some parts of the flow field further downstream.

The velocity field due to a vortex can be evaluated by means of the Biot-Savart law. In this manner one can evaluate, for instance, the velocity field of a vortex extending along the x axis from negative to positive infinity; in this manner one obtains the so-called potential vortex. If one extends the integration from negative infinity to a finite value $x = x_0$, then it seems as if one violates the condition div curl $\vec{u} = 0$ at this point because one does not continue the vortex. Actually one creates a field in which vortex lines, equally distributed over all directions, emerge from the point $x = x_0$ in the same manner as stream lines emerge from a source. If one fails to take h_y into account in part of the wake for downstream, one no longer expresses the vorticity present in that part of the wake but introduces vorticity sources at the cut-off line, which introduces vorticity in the entire flow field. The effect of these vortices at a distance is weak because the potential outside of the wake is zero. The integral $\int h_y dy$ evaluated across the wake is zero; therefore, one has for every vortex source also a vortex sink in a vicinity smaller than the vortex width. In any case the final steady state in the three-dimensional case with a truncated wake is not the same as for a non-truncated wake.

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2. Maxwell Blair, Program KGG2D Application Manual Flight Dynamics Laboratory. AFWAL-TM-86-216-FIBR.

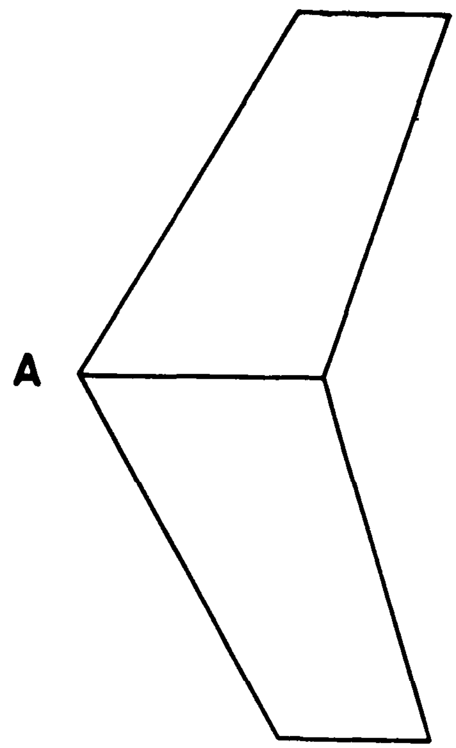

Figure 1. Planform.

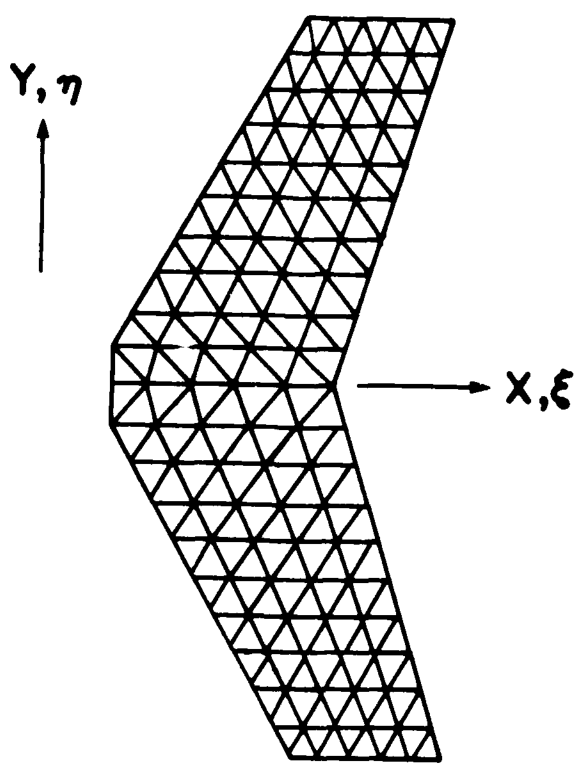

Figure 2. Modified Planform, Subdivision into Triangles.

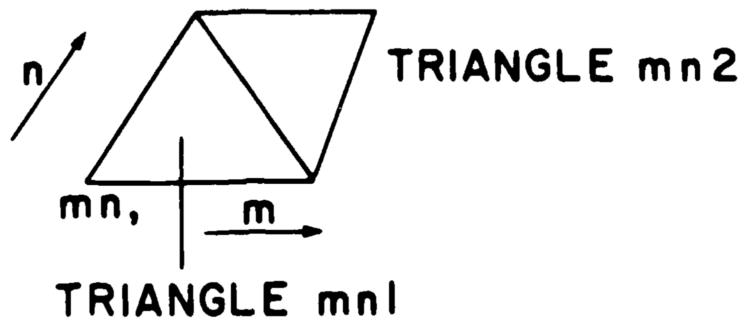

Figure 3. Numbering of Triangles.

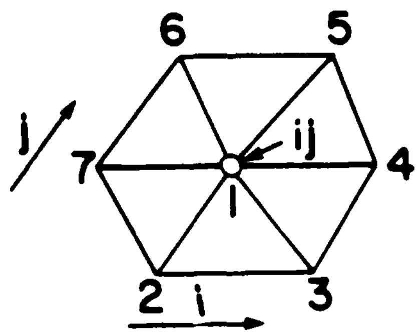

Figure 4. Numbering of the Vertices at and Adjacent to the Control Point ij.

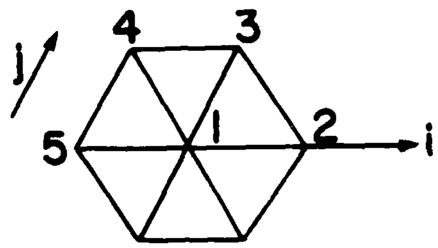

Figure 5. Numbering of the Corners at and Adjacent to a Control Point on the Axis of Symmetry (j=0).

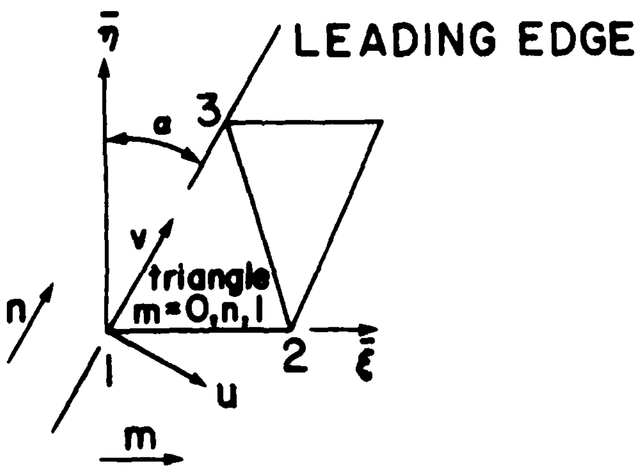

Figure 6. Triangle m=0, n, 1 (at the leading edge). Sweep Angle, $\bar{\xi}$ $\bar{\eta}$ and uv Systems.

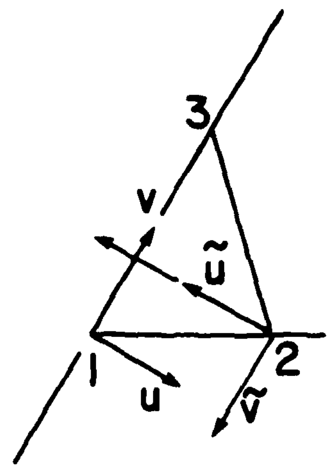

Figure 7. Same Triangle as Figure 6, uv and $\tilde{u}\ \tilde{v}$ Systems.

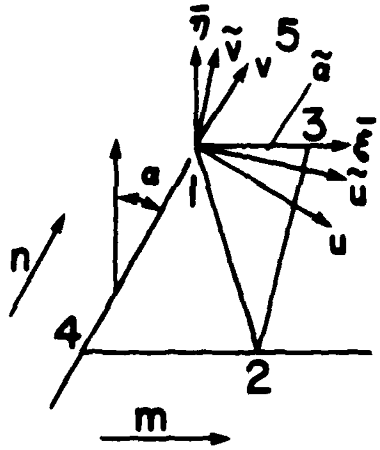

Figure 8. Triangle m=0, n, 2 at the Leading Edge, $\xi\eta$, uv, and $\tilde{u}\tilde{v}$ Systems.

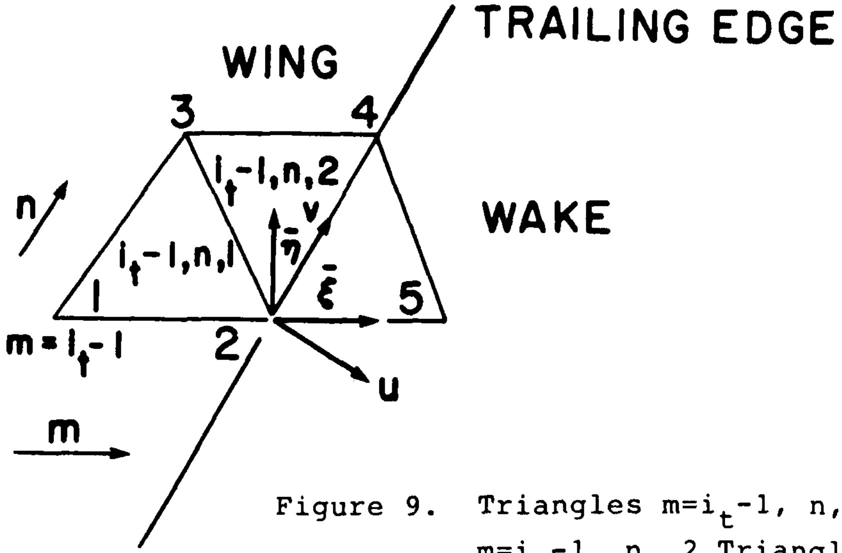

Figure 9. Triangles m=i_t-1, n, 1 and m=i_t-1, n, 2 Triangle in the Wake $\bar{\xi}\bar{\eta}$ and uv Systems.

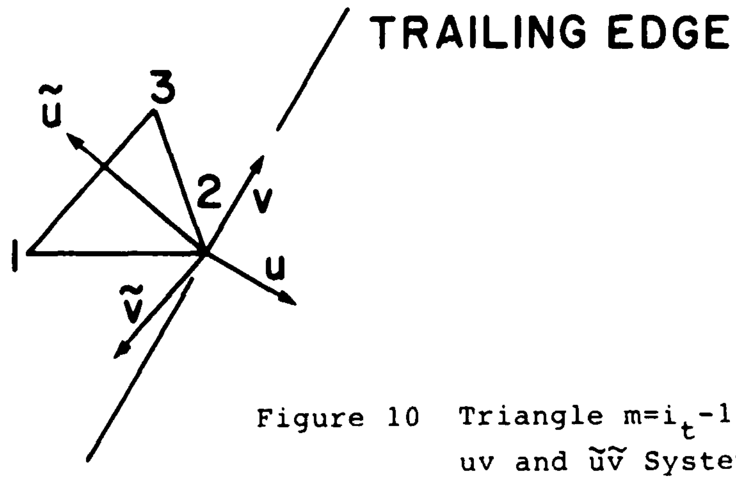

Figure 10 Triangle m=i_t-1, n, 1 uv and $\tilde{u}\tilde{v}$ Systems.

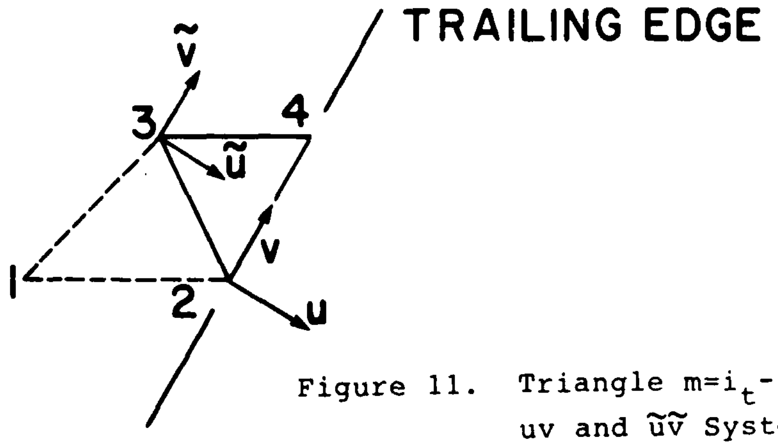

Figure 11. Triangle $m=i_t-1$, n, 2 uv and $\tilde{u}\tilde{v}$ Systems.

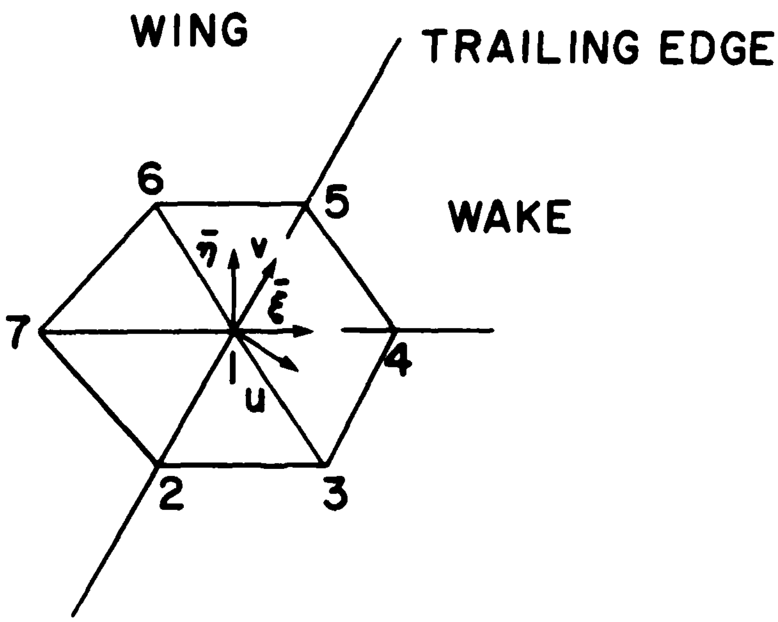

Figure 12. Vicinity of a Control Point at the Trailing edge ($i=i_t$) $i=i_t$, $\bar{\xi}$ $\bar{\eta}$ and uv Systems.

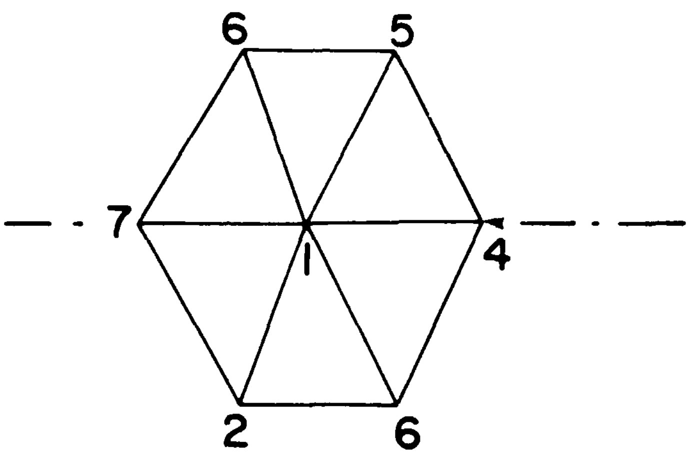

Figure 13. Vicinity of a Control Point in the Interior at the Axis of Symmetry.

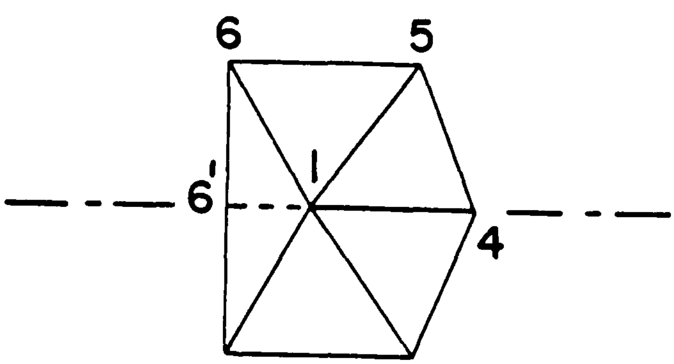

Figure 14. Control Point at the Axis of Symmetry Next to the Leading Edge.

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